DYNAMICS OF A PARTICLE

FOR

HONOURS AND POST-GRADUATE STUDENTS

OF

All Indian Universities and for various Competitive Examinations

By

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PREFACE TO THE FIRST EDITION

Due to my pre-occupation with my other thirty-five books I could not bring out this book earlier as promised and I regret very much for the inconvenience that was caused to my readers.

The present book is on the same pattern and style as my other books with which the students are very well acquainted. Mostly students feel Mechanics paper to be the toughest and keeping this fact in view I have explained every point to the minutest detail and I am sure that the students will be able to understand the subject without any difficulty. The book on Integral Calculus for M. Sc. is also out along with this book and I hope but do not promise to bring out the books on Statics and Spherical Harmonics within this session.

Any suggestions for improvement of the book from any quarter will be highly appreciated.

315, Chhipi Tank,
Meerut.
July 1963.

M. L. KHANNA
Associate Professor

PREFACE TO THE THIRD EDITION

The subject matter has been revised and a few questions have been added here and there. Latest papers of various universities have been added in the end.

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Meerut College, Meerut.
Oct. 1966.
Phone 2217.

M. L. KHANNA

Where is What ?

Note. Here below we give the contents of Dynamics for B. Sc. and Dynamics for Intermediate or B. Sc. Pt. I students.

Dynamics for B. Sc. **(1) Kinematics.** Radial, Transverse, Tangential, normal, angular velocities and accelerations. **(2) Rectilinear Motion.** Simple Harmonic motion, Hooke's Law, Horizontal elastic strings, vertical elastic strings, Universal law of gravitation, Motion under other laws of forces. **(3) Projectiles.** **(4) Constrained motion** Motion inside a vertical circle, Motion outside a vertical circle, Motion on a cycloid, Simple Pendlum **(5) Impact.** Direct and oblique impact of two spheres, Impact of a sphere on a fixed plane. **(6) Relative velocity.** **(7) Impulse work and Energy.** **(8) University Papers.**

Dynamics for Inter. (1) Motion in a straight line. (2) Motion under gravity. (3) Laws of motion. (4) Impulse work and energy. (5) Motion of particles connected by a string. (6) Projectiles. (7) Impact of elastic bodies. (8) Relative velocity. (9) Test Papers.

LISTS OF IMPORTANT FORMULAE

Chapter I **Central Forces.**

(1) $h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P.$ (2) $P=\frac{h^2}{p^3}\frac{dp}{dr}.$

(3) $v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right].$ (4) $h=r^2\frac{d\theta}{dt}=vp.$

(5) At an apse $\frac{du}{d\theta}=0.$ (6) $p=r\sin\phi.$

(7) $\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$

(8) Differential equation of the path under radial and transverse accelerations $h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P-\frac{T}{u}\frac{du}{d\theta}$,

where P is radial acceleration towards centre and T is transverse.

Chapter II. **The Inverse Square Law.**

(1) If $P=\frac{\mu}{r^2}$, then $v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right)$, if orbit be an ellipse ;

$v^2=\mu\left(\frac{2}{r}+\frac{1}{a}\right)$, if the orbit be a hyperbola ; and

$v^2=\frac{2\mu}{r}$ if the orbit be a parabola ; where $h^2=\mu l$

l is semi-latus rectum$=\frac{b^2}{a}$.

(2) Square of the periodic time of an elliptical orbit varies as the cube of the major axis.

(3) Some important results of ellipse.
See P. 112 and remember all the results from 1 to 7.

Chapter III. **Constrained Motion.**

(1) $\tan\psi=\dfrac{dy}{dx}$; $\therefore$ $\sin\psi=\dfrac{dy}{ds}$ and $\cos\psi=\dfrac{dx}{ds}$.

(2) **Equation of Energy.** Change in K. E.=work done, *i e.* $\frac{1}{2}mv^2-\frac{1}{2}mu^2=f(x, y)-f(x_0, y_0)$.

(3) **Principle of Energy.** The sum of kinetic and potential energies of the particle is constant throughout the motion.

(4) $\displaystyle\int e^{ax}\sin bx=\frac{e^{ax}}{(a^2+b^2)}(a\sin bx-b\cos bx)$,

$\displaystyle\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{(a^2+b^2)}(a\cos bx+b\sin bx)$.

(5) $\tan\phi=r\dfrac{d\theta}{dr}$; $\therefore$ $\sin\phi=r\dfrac{d\theta}{ds}$, $\cos\phi=\dfrac{dr}{ds}$.

(6) $\rho=\dfrac{(1-y'^2)^{3/2}}{y''}=\dfrac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}=r\cdot\dfrac{dr}{dp}$.

(7) Intrinsic equation of cycloid is $s=4a\sin\psi$.

(8) $\displaystyle\int e^{-2\mu\psi}\cos\psi\,(\mu\cos\psi-\sin\psi)\,d\psi$ is done by putting

$e^{-\mu\psi}(\mu\cos\psi-\sin\psi)=z$; $\therefore$ $e^{-\mu\psi}\cos\psi\,d\psi=-\dfrac{dz}{1+\mu^2}$.

[P. 223]

(9) Velocity in the case of cycloidal motion is given by

$v^2e^{-2\mu\psi}=-\dfrac{4ag}{1+\mu^2}e^{-2\mu\psi}(\mu\cos\psi-\sin\psi)^2+C.$

[P. 224]

Chapter IV. **Motion in a Resisting Medium.**

(1) **Resistance varying as square of velocity.**

(a) **Vertically downward motion.**

1. $\dfrac{d^2x}{dt^2}=g\left(1-\dfrac{v^2}{V^2}\right)$. 2. $v^2=V^2(1-e^{-2gx/V^2})$.

3. $v=V\tanh\frac{g}{V}t$. 4. $x=\frac{V^2}{g}\log\cosh\frac{g}{V}t$,

[P. 245]

where V is terminal velocity and $\frac{k}{g}=\frac{1}{V^2}$.

(b) **Vertically upward motion**

1. $\frac{d^2x}{dt^2}=-\frac{g}{V^2}(V^2+v^2)$, 2. $\frac{2g}{V^2}x=\log\frac{V^2+u^2}{V^2+v^2}$.

3. $t=\frac{V}{g}\left(\tan^{-1}\frac{u}{V}-\tan^{-1}\frac{v}{V}\right)$

where V is the terminal velocity and u the velocity of projection. [P. 247]

(2) **Resistance varying as velocity for a projectile.**

(i) $\frac{d^2x}{dt^2}=-k\frac{dx}{dt}$. (ii) $\frac{d^2y}{dt^2}=-g-k\frac{dy}{dt}$.

(iii) $\frac{dx}{dt}=u\cos\alpha . e^{-kt}$. (iv) $g+k\frac{dy}{dt}=(g+ku\sin\alpha)e^{-kt}$

(v) $x=\frac{u\cos\alpha}{k}(1-e^{-kt})$.

(vi) $gt+ky=\frac{g+ku\sin\alpha}{k}(1-e^{-kt})$.

(vii) $y=\frac{g}{k^2}\log\left(1-\frac{kx}{u\cos\alpha}\right)+\frac{x}{ku\cos\alpha}(g+ku\sin\alpha)$.

[P. 271-73]

Chapter V. **Hodograph.**

Acceleration at any point P on the curve is equal to the corresponding velocity at the point Q on the hodograph.

In order to find hodograph we should find a (v, ψ) relation, *i.e.* $v=f(\psi)$ and in it replace v by r/λ and ψ by θ.

Central Orbit. The hodograph of a central orbit is the reciprocal of the orbit with respect to the centre of force S turned through a right angle about S.

Rule. Take (r_1, θ_1) the foot Y of the perpendicular from centre of force and find a relation between (r_1, θ_1). Replace r_1 by k^2/r_1 and generalise and then you will get the locus of P' on SY produced such that $SY.SP'=k^2$. This is known as reciprocal of given orbit. Now turn it through a right angle to get the hodograph.

Chapter VI. **Revolving Curves.**

Rule. Introduce along the radius vector a force equal to $mr\omega^2$ in addition to the given forces and take the reaction to be R' and now treat the question as if the curve were fixed. It is not necessary to write the radial and transverse equations of motion once the curve is reduced to rest. You may resolve the forces in any manner you like *i e.* you may even write tangential and normal equations of motion. Another point to be noted is that the value of R' will not be true normal reaction. The true value of $R=R'+2m\omega v$, where v is the velocity of the bead w.r.t. the tube (Note) and will be $+$ive if its direction is the same as that in which the tube revolves. If however the direction of the bead and that of the tube be opposite, *i e.* one moves clockwise and the other anti-clockwise, then v will be $-$ive and in that case true normal reaction $R=R'+2m\omega(-v)$.



CHAPTER I

CENTRAL FORCES

§ 1. **Definitions :** Students must have read in B. Sc. classes (Refer author's Dynamics for B. Sc. classes) that when a particle moves along a curve then it has the following velocities and accelerations.

Radial Velocity $=\dfrac{dr}{dt}=\dot{r}$.

Transverse Velocity $=r\dfrac{d\theta}{dt}=r\dot{\theta}$.

Radial Acceleration $=\dfrac{d^2r}{dt^2}-r\left(\dfrac{d\theta}{dt}\right)^2=\ddot{r}-r\dot{\theta}^2$

Transverse Acceleration $=2\dfrac{dr}{dt}\dfrac{d\theta}{dt}+r\dfrac{d^2\theta}{dt^2}=\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)$.

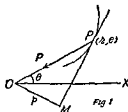
These components of velocity and acceleration are +ive in the sense of **r** increasing in the case of radial and in the sense of θ increasing in the case of transverse.

Central Force : A force which is directed towards a fixed point is called central force and the path described by the particle moving along a plane curve under a central force is called central orbit and the fixed point is called the centre of force. The fixed point O, *i.e.* the centre of force is taken as the pole (origin) and a horizontal line OX through O as the initial line. The position of the particle P at any time t is denoted by its polar co-ordinates (r, θ) referred to O as pole and OX as initial line.

§ 2. *A particle moves in a plane curve with an acceleration P which is always directed to a fixed point O in the plane. Find the differential equation of its path.*

[Punjab 58, 55, 54, 53 ; Nagpur 54 (S) ; I. A. S. 55, 53, 50, 48 ; Osmania 63 ; Raj. 65; Agra 66, 63, 58, 55, 53, 49, 47]

Polar Form. Since the acceleration is directed towards the fixed point O (taken as pole) hence the particle has only radial acceleration and transverse acceleration therefore is zero. Again as the acceleration is directed towards O *i.e.* in the sense of r decreasing, we have the following equations of motion.



$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=-P, \quad \text{(Note } -P) \qquad \ldots(1)$$

and
$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=0. \qquad \ldots(2)$$

Integrating (2), we get

$$\mathbf{r^2\frac{d\theta}{dt}=constant=h \ (say)} \qquad \ldots(3)$$

$$\therefore \quad \frac{d\theta}{dt}=\frac{h}{r^2}=hu^2 \quad \text{where } u=\frac{1}{r}. \qquad \ldots(4)$$

Now we have to find the value of $\dfrac{d^2r}{dt^2}$.

From the relation $u=\dfrac{1}{r}$ *i.e.* $r=\dfrac{1}{u}$, we have on differentiation,

$$\frac{dr}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}=-\frac{1}{u^2}\frac{du}{dt}\cdot\frac{d\theta}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}\cdot hu^2 \text{ from (4)}$$

or
$$\frac{dr}{dt}=-h\frac{du}{d\theta} \qquad \ldots(5)$$

Differentiating both sides of (5) w. r. t. t, we get

$$\frac{d^2r}{dt^2}=\frac{d}{dt}\left(-h\frac{du}{d\theta}\right)=-h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\cdot\frac{d\theta}{dt}$$

$$=-h\frac{d^2u}{d\theta^2}(hu^2) \text{ by } (4)=-h^2u^2\frac{d^2u}{d\theta^2}. \qquad \ldots(6)$$

Now putting for $\frac{d^2r}{dt^2}$ and $\frac{d\theta}{dt}$ from (6) and (4) respectively in (1), we get

$$-h^2u^2\frac{d^2u}{d\theta^2}-r.h^2u^4=-P. \quad \text{Put } r=\frac{1}{u}.$$

$$\therefore \quad \mathbf{\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}} \qquad \ldots(7)$$

Above gives us the required differential equation of the central orbit in polar co-ordinates.

Pedal Form. If p be the length of perpendicular from pole O on the tangent at P, then we know from differential calculus that $\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$. **(Agra 65 ; Mysore 66)**

As $u=\frac{1}{r}$ $\therefore$ $\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$ or $\left(\frac{du}{d\theta}\right)^2=\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$.

Hence $$\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2. \qquad \ldots(8)$$

Differentiating both sides of above w. r. t. θ, we get

$$-\frac{2}{p^3}\frac{dp}{d\theta}=2u\frac{du}{d\theta}+2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}$$

or $$-\frac{1}{p^3}\frac{dp}{d\theta}=\frac{du}{d\theta}\left(u+\frac{d^2u}{d\theta^2}\right)=-\frac{1}{r^2}\frac{dr}{d\theta}\cdot\frac{P}{h^2u^2} \text{ from (7)}$$

$$=-\frac{P}{h^2}\frac{dr}{d\theta}. \quad \because \quad u=\frac{1}{r}.$$

$$\therefore \quad \mathbf{\frac{h^2}{p^3}\cdot\frac{dp}{dr}=P.} \qquad \ldots(9)$$

Above is the p, r relation *i.e.* pedal equation of the central orbit. Equations (7) and (9) will be used frequently and students are advised to commit them to memory. We shall use them in the following type of problems.

(a) When we are given the law of the central force, to determine the orbit.

In this case P *is known as a function of* r *i.e. of* u *and* equation (7) becomes a differential equation of 2nd order whose solution will give us a relation in u and θ *i.e.* r and θ and two constants whose values will be obtained from initial conditions and hence we will have the equation to the orbit.

(b) When we are given the orbit, to determine the law of the central force.

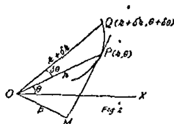
In this case r, θ relation *i.e.* u, θ relation is known and calculating $\frac{d^2u}{d\theta^2}$ and putting in (7), we get P giving the law of central force.

§ 3. Aerial velocity and linear velocity in a central orbit.

When a particle moves along a plane curve, the rate of change of the area traced out by the radius vector joining the particle to the centre of force is called the aerial velocity of the particle.

Expression for aerial velocity.

(a) Let P (r, θ) be the position of the particle at any time t and Q $(r+\delta r, \theta+\delta\theta)$ be its position at time $(t+\delta t)$. Thus during the elementary interval δt the area swept out by OP is $\triangle OPQ$. Hence the aerial velocity of the particle is



$$\underset{\delta t\to 0}{\text{Lt}}\ \frac{\triangle OPQ}{\delta t}=\underset{\delta t\to 0}{\text{Lt}}\ \frac{1}{2}\frac{r\,(r+\delta r)\sin\delta\theta}{\delta t}$$

$$=\underset{\delta t\to 0}{\text{Lt}}\cdot\frac{1}{2}r\,(r+\delta r)\,\frac{\sin\delta\theta}{\delta\theta}\cdot\frac{\delta\theta}{\delta t}=\frac{1}{2}r^2\frac{d\theta}{dt}.$$

But from result 3 P. 2, we have

$$r^2\frac{d\theta}{dt}=\text{constant}=h.$$

$\therefore$ Aerial velocity $=\frac{1}{2}r^2\frac{d\theta}{dt}=\frac{1}{2}h$ *i. e.* constant. ...(1)

Hence we have the following theorem.

In every central orbit the sectorial area traced out by the radius vector to the centre of force increases uniformly per unit of time. **(Vikram 1965; Nagpur 52; Agra 48)**

(b) Aerial velocity

$$=\underset{\delta t\to 0}{\text{Lt}}\ \frac{\triangle OPQ}{\delta t}$$

$$=\underset{\delta t\to 0}{\text{Lt}}\ \frac{1}{2}\ \frac{\text{chord } PQ\times\perp \text{ from } O \text{ on } PQ}{\delta t}$$

$$=\underset{\delta t\to 0}{\text{Lt}}\ \frac{1}{2}\cdot\frac{\delta c}{\delta s}\cdot\frac{\delta s}{\delta t}.\text{perp. from } O \text{ on } PQ.$$

Now as $\delta t\to 0$, $Q\to P$ and PQ tends to be the tangent at P and hence perpendicular from O on PQ will tend to P in limit. Also we know that $\text{Lt}\,\frac{\delta c}{\delta s}=1$ and $\text{Lt}\,\frac{\delta s}{\delta t}=\frac{ds}{dt}$ = linear velocity v of the particle.

$\therefore$ Aerial velocity $=\frac{1}{2}.v.p.=\frac{1}{2}h$ [by part (a) (1)].

$$\therefore\quad h=vp \quad\text{or}\quad v=\frac{h}{p}. \qquad \ldots(2)$$

Hence, we have the following theorem :—

In every central orbit the linear velocity at a point P of its path var[ies] ... e centre upon the ...

(Vikram 1965; Nagpur 52, Agra 48)

Cor. We know that $\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2$ and $v=\frac{h}{p}$.

$$\therefore \quad v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]. \qquad \ldots(3)$$

[Nagpur 55 (Supp.)]

(c) If we suppose that P is (x, y) and Q $(x+\delta x, y+\delta y)$, then the area of triangle OPQ

$$=\tfrac{1}{2}[x_1y_2-x_2y_1]=\tfrac{1}{2}[x(y+\delta y)-y(x+\delta x)]$$
$$=\tfrac{1}{2}[x\,\delta y-y\,\delta x].$$

$$\therefore \quad \text{Aerial velocity}=\text{Lt}\,\frac{\triangle OPQ}{\delta t}=\text{Lt}\,\frac{1}{2}\left[x\frac{\delta y}{\delta t}-y\frac{\delta x}{\delta t}\right]$$
$$=\tfrac{1}{2}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right).$$

Hence from (a), (b), (c), we have

twice the aerial velocity

$$=r^2\frac{d\theta}{dt}=vp=x\frac{dy}{dt}-y\frac{dx}{dt}=h$$

or **the rate of description of sectorial area by the radius vector = constant** $=\frac{h}{2}$.

(d) In part (b), we have proved that vp=constant.

$\therefore$ $(mv)\,p$=constant.

Above shows that the moment of momentum of the particle about the centre of force is constant.

(e) Time to describe the orbit or any part of it.

From the relation $r^2\frac{d\theta}{dt}=h$, we have

$$r^2\,d\theta=h\,dt.$$

Integrating above between proper limits, we have

$$ht=\int_{\alpha}^{\beta} r^2\,d\theta.$$

(f) Converse. *If the radius vector to a point which is describing an orbit sweeps out equal areas in equal intervals of time, then the acceleration must be directed towards the centre.*

We are given that aerial velocity of the particle is constant.

$$\therefore \quad \tfrac{1}{2}r^2\frac{d\theta}{dt}=\text{constant} \quad \text{or} \quad r^2\frac{d\theta}{dt}=\text{constant} \quad \ldots(1)$$

Now let us suppose that the transverse component of acceleration be T ; then

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=T$$

or

$$\frac{1}{r}\frac{d}{dt}(\text{constant})=T \quad \text{or} \quad 0=T \text{ by (1).}$$

Above shows that transverse acceleration is zero ; hence it must be directed towards the centre.

§ 4. Law of force, velocity and periodic time when the orbit is an ellipse.

A particle moves in an ellipse under a force which is always directed towards the focus, to find the law of force and velocity at any point of its path. (I. A. S. 1932, 48 ; Agra 58, 57, 54, 51, 46 ; Punjab 52)

Law of force. We know that polar equation of the ellipse referred to focus as pole is

$$\frac{l}{r}=1+e\cos\theta \quad \text{or} \quad u=\frac{1+e\cos\theta}{l}. \quad \ldots(1)$$

$$\therefore \quad \frac{du}{d\theta}=-\frac{e}{l}\sin\theta \text{ and } \frac{d^2u}{d\theta^2}=-\frac{e}{l}\cos\theta. \quad \ldots(2)$$

Now the ellipse is described under a force which is directed towards the focus *i. e.* pole ; hence from equation (7) of § 2, we have

$$\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}$$

or $$\frac{-e}{l}\cos\theta+\frac{1}{l}+\frac{e}{l}\cos\theta=\frac{P}{h^2u^2}.$$

$$\therefore\quad P=\frac{h^2}{l}u^2=\frac{h^2}{l}\cdot\frac{1}{r^2}=\frac{\mu}{r^2} \qquad \ldots(3)$$

where $\mu=\dfrac{h^2}{l}$ or $h=\sqrt{(\mu . l)}$

or $h=[\mu\ .\ (\text{semi-latus rectum})]^{1/2}$.

Hence the acceleration varies inversely as the square of the distance of the particle from the focus

Velocity at any point. We know from cor. of § 3 that in a central orbit velocity at any point is given by

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right].$$

$$\therefore\quad v^2=\mu l\left[\frac{1+2e\cos\theta+e^2\cos^2\theta}{l^2}+\frac{e^2\sin^2\theta}{l^2}\right]$$

by (1), (2)

or $$v^2=\mu\left(\frac{1+2e\cos\theta+e^2}{l}\right)=\mu\left[\frac{2(1+e\cos\theta)}{l}-\frac{1-e^2}{l}\right]$$

or $$v^2=\mu\left[\frac{2}{r}-\frac{1}{a}\right] \qquad \ldots(4)$$

$$\because\quad \frac{l}{r}=1+e\cos\theta \text{ and } l=\frac{b^2}{a}=\frac{a^2(1-e^2)}{a}=a(1-e^2)$$

$2a$ being the length of major axis of the ellipse.

From above we conclude that the velocity at any point of the path depends only on r, i.e. the distance of the point from the focus and that it is independent of the direction of motion.

Periodic Time.

If T be the total time of describing the whole arc of the ellipse, then we know that the rate of description of area is $\frac{1}{2}h$, which is constant.

Areal Velocity = $\frac{1}{2}h$ (Constant)

$\therefore \quad \frac{1}{2}h \times T$ = area of ellipse $= \pi ab$.

$$\therefore \quad T = \frac{2\pi ab}{h} = \frac{2\pi ab}{\sqrt{(\mu l)}} = \frac{2\pi ab}{\sqrt{\left(\mu \cdot \frac{b^2}{a}\right)}} = \frac{2\pi a^{3/2}}{\sqrt{\mu}} \qquad \ldots(5)$$

Alternative Method. Pedal form.

We know that the pedal equation of the ellipse is

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1. \qquad \ldots(1)$$

[referred to focus as pole]

Also from equation (9) § 2, we have

$$\frac{h^2}{p^3} \frac{dp}{dr} = P. \qquad \ldots(2)$$

Differentiating (1) w. r. t. r, we get

$$\frac{-2b^2}{p^3} \frac{dp}{dr} = \frac{-2a}{r^2}$$

or
$$\frac{1}{p^3} \frac{dp}{dr} = \frac{a}{b^2 r^2}.$$

Hence from (2), $h^2 \cdot \frac{a}{b^2 r^2} = P$.

But $\quad \frac{b^2}{a} = l$ = semi-latus rectum.

$\therefore \quad P = \frac{h^2}{lr^2} = \frac{\mu}{r^2}$, say where $\mu = \frac{h^2}{l}$.

Again we know that $vp = h$ [§ 3 (b)].

$$\therefore \quad v^2 = \frac{h^2}{p^2} = \frac{\mu l}{b^2}\left[\frac{2a}{r} - 1\right] = \frac{\mu}{a}\left[\frac{2a}{r} - 1\right], \because l = \frac{b^2}{a}$$

or
$$v^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right].$$

Working rule.

(i) If the polar equation of the curve be given, then

put $u=\frac{1}{r}$ and from the given equation find $\frac{d^2u}{d\theta^2}$ then

$$P=h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right).$$

(ii) If the pedal equation of the curve be given, then use the corresponding formula

$$P=\frac{h^2}{p^3}\frac{dp}{dr}.$$

(iii) $$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right].$$

Ex. 1. *A particle describes the curve $r^n=a^n \cos n\theta$ under a force P to the pole. Find the law of force.*

1st Method. Putting $u=\frac{1}{r}$, we get $a^n u^n \cos n\theta=1$.

Taking log, we get $n \log a+n \log u+\log \cos n\theta=0$.

Differentiating both sides w. r. t. θ, we get

$$n.\frac{1}{u}\frac{du}{d\theta}=-\frac{1}{\cos n\theta}\times-\sin n\theta . n\;;\;\therefore\;\frac{du}{d\theta}=u \tan n\theta \quad \ldots(1)$$

$$\therefore\;\frac{d^2u}{d\theta^2}=\frac{du}{d\theta}\tan n\theta+u.n \sec^2 n\theta$$

$$=u \tan^2 n\theta+u.n \sec^2 n\theta, \text{ by (1)}.$$

Now $$P=h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)$$

$$=h^2u^2\left[u+u \tan^2 n\theta+u.n \sec^2 n\theta\right]$$

or $$P=h^2u^2\left[u \sec^2 n\theta+un \sec^2 n\theta\right]$$

$$=h^2(n+1)\,u^3 \sec^2 n\theta$$

or $$P=h^2(n+1)\,u^3.u^{2n}.a^{2n}=h^2(n+1).\frac{a^{2n}}{r^{2n+3}} \quad \ldots(2)$$

Hence the law of force is varying inversely as $(2n+3)$rd power of the distance from the pole.

2nd Method. Pedal form.

$$r^n=a^n \cos n\theta.$$

Taking log of both sides, $n \log r = n \log a + \log \cos n\theta$.

Differentiating w. r. t. θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{1}{\cos n\theta} \times -\sin n\theta . n$$

or $$\cot \phi = -\tan n\theta = \cot\left(\frac{\pi}{2} + n\theta\right).$$

$$\therefore \quad \phi = \frac{\pi}{2} + n\theta.$$

Now we know that

$$p = r \sin \phi = r \cos n\theta = r\ \frac{r^n}{a^n} = \frac{r^{n+1}}{a^n}.$$

Hence the pedal equation is $p = \dfrac{r^{n+1}}{a^n}$. ...(3)

We know that $P = \dfrac{h^2}{p^3}\dfrac{dp}{dr} = \dfrac{h^2}{p^3} . (n+1)\ \dfrac{r^n}{a^n}$ by (3)

or $$P = h^2 (n+1) \frac{r^n}{a^n} . \frac{a^{3n}}{r^{3n+3}} = h^2 (n+1) \frac{a^{2n}}{r^{2n+3}} \quad ...(4)$$

Above is same as (2).

Particular Cases.

(i) If $n = \frac{1}{2}$, then the curve becomes $r^{1/2} = a^{1/2} \cos \dfrac{\theta}{2}$ or $r = a \cos^2 \dfrac{\theta}{2}$ or $r = \dfrac{a}{2}(1 + \cos \theta)$ which represents a cardioid $\quad 2n+3 = 2\left(\frac{1}{2}\right)+3 = 4$.

Hence in a cardioid the force varies inversely as fourth power of the distance from the pole. Also its pedal equation from (3) on putting $n = \frac{1}{2}$ and squaring is $ap^2 = r^3$.

Proceeding directly.

$$r = \frac{a}{2}(1 + \cos \theta) \text{ or } 1 = \frac{a}{2} . u (1 + \cos \theta) \text{ or } 1 = a . u . \cos^2 \frac{\theta}{2}. \quad ...(1)$$

Taking log and differentiating, we get

$$0=0+\frac{1}{u}\cdot\frac{du}{d\theta}+\frac{-\sin\theta}{1+\cos\theta} \text{ or } \frac{du}{d\theta}=u\tan\frac{\theta}{2}. \quad \ldots(2)$$

$$\therefore \quad \frac{d^2u}{d\theta^2}=\frac{du}{d\theta}\tan\frac{\theta}{2}+u\cdot\tfrac{1}{2}\sec^2\frac{\theta}{2}=u\tan^2\frac{\theta}{2}+\tfrac{1}{2}u\sec^2\frac{\theta}{2}, \text{ by}(2)$$

$$\therefore \quad P=h^2u^2\left[u+\frac{d^2u}{d\theta^2}\right]=h^2u^2\left[u+u\tan^2\frac{\theta}{2}+\tfrac{1}{2}u\sec^2\frac{\theta}{2}\right]$$

or $$P=h^2u^2\left[u\sec^2\frac{\theta}{2}+\tfrac{1}{2}u\sec^2\frac{\theta}{2}\right]$$

$$=h^2\cdot u^2\cdot u\sec^2\frac{\theta}{2}\,[1+\tfrac{1}{2}]$$

or $$P=\tfrac{3}{2}h^2u^3\cdot au \text{ by } (1)=\tfrac{3}{2}\,\frac{ah^2}{r^4}.$$

2nd Method. The pedal equation can be easily shown to be

$$ap^2=r^3$$

or $$p=\frac{r^{3/2}}{\sqrt{a}};\quad \therefore \quad \frac{dp}{dr}=\tfrac{3}{2}\sqrt{\frac{r}{a}}.$$

$$\therefore \quad P=\frac{h^2}{p^2}\frac{dp}{dr}=\frac{h^2\cdot a^{3/2}}{r^{3/2}}\cdot\frac{3}{2}\cdot\sqrt{\frac{r}{a}}=\frac{3}{2}\frac{h^2\cdot a}{r^4}.$$

(ii) If $n=-\frac{1}{2}$, curve becomes $r^{-1/2}=a^{-1/2}\cos\left(-\frac{\theta}{2}\right)$

or $$r=\frac{a}{\cos^2\theta/2}=\frac{2a}{1+\cos\theta} \text{ or } \frac{2a}{r}=1+\cos\theta.$$

Above represent a parabola and $2n+3=2\,(-\frac{1}{2})+3=2$.

Hence in a parabola the force varies inversely as square of the distance from the pole. Also its pedal equation from (3), on putting $n=-\frac{1}{2}$ and squaring is $p^2=ar$.

(iii) If $n=2$, then the curve becomes $r^2=a^2\cos 2\theta$, which is Lemniscate of Bernoulli and $2n+3=2\,(2)+3=7$.

Hence in Lemniscate of Bernoulli the force varies inversely as 7th power of the distance from the pole. Also its pedal equation is $pa^2=r^3$ from (3) on putting $n=2$.

(iv) If $n=-2$, the curve becomes $r^{-2}=a^{-2}\cos(-2\theta)$

or $$r^2\cos 2\theta=a^2 \quad \text{or} \quad x^2-y^2=a^2,$$

which represents a rectangular hyperbola

and $$2n+3=2(-2)+3=-1.$$

Hence in a rectangular hyperbola the force varies inversely as r^{-1} or directly as r but since $(n+1)=-2+1=-1$, hence we should say that force varies as $-r$. Also its pedal equation is $pr=a^2$.

Proceeding directly,

$$r^2\cos 2\theta=a^2, \quad \text{or} \quad a^2u^2=\cos 2\theta.$$

Taking log and differentiating, we get

$$0+\frac{2}{u}\frac{du}{d\theta}=\frac{1}{\cos 2\theta}(-2\sin 2\theta) \quad \text{or} \quad \frac{du}{d\theta}=-u\tan 2\theta.$$

$$\therefore \quad \frac{d^2u}{d\theta^2}=-\left[\frac{du}{d\theta}\tan 2\theta+u.2\sec^2 2\theta\right].$$

$$=-[-u\tan^2 2\theta+u.2\sec^2 2\theta].$$

$$\therefore \quad P=h^2u^2\left[u+\frac{d^2u}{d\theta^2}\right]=h^2u^2[u+u\tan^2 2\theta-2u\sec^2 2\theta]$$

$$=h^2u^2.u\sec^2 2\theta[1-2]$$

$$=-h^2u^2.\frac{1}{a^4u^4}=-\frac{h^2}{a^4}\frac{1}{u}=-\frac{h^2}{a^4}r.$$

Hence force varies directly as r.

2nd Method. The pedal equation can be easily shown to be $$pr=a^2$$

or $$p=\frac{a^2}{r}; \quad \therefore \quad \frac{dp}{dr}=-\frac{a^2}{r^2}.$$

$$\therefore \quad P=\frac{h^2}{p^3}\cdot\frac{dp}{dr}=h^2.\frac{r^3}{a^6}\left(-\frac{a^2}{r^2}\right)=-\frac{h^2}{a^4}.r \text{ etc.}$$

(v) If $n=1$, then the curve becomes $r=a\cos\theta$ which represents a circle passing through the pole and whose diameter is along initial line and of length a. Also $2n+3=5$.

Hence in a circle the force varies inversely as 5th power of the distance from the pole. Also its pedal equation is $ap=r^2$.

(vi) If $n=-n$, then the curve becomes

$$r^{-n}=a^{-n}\cos(-n\theta)$$

or $\quad r^n\cos n\theta=a^n$ and $2n+3=-2n+3. \quad \because \quad n=-n.$

Hence in the curve $r^n\cos n\theta=a^n$ the force varies inversely as $(-2n+3)$rd power or directly as $(2n-3)$rd power of the distance from the pole.

Proceeding directly.

$$r^n\cos n\theta=a^n \quad \text{or} \quad a^nu^n=\cos n\theta.$$

Taking log and differentiating, we get

$$n.\frac{1}{u}.\frac{du}{d\theta}=-n\frac{\sin n\theta}{\cos n\theta}; \quad \therefore \quad \frac{du}{d\theta}=-u\tan n\theta.$$

$$\therefore \quad \frac{d^2u}{d\theta^2}=-\frac{du}{d\theta}\tan n\theta-un\sec^2 n\theta$$

$$=u\tan^2 n\theta-un\sec^2 n\theta.$$

$$\therefore \quad P=h^2u^2\left[u+\frac{d^2u}{d\theta^2}\right]=h^2u^2\left[u\sec^2 n\theta-un\sec^2 n\theta\right]$$

$$=h^2u^3.\sec^2 n\theta\,(1-n)=h^2u^3.\frac{1}{a^{2n}.u^{2n}}(1-n)$$

or $$P=\frac{h^3}{a^{2n}}(1-n).u^{3-2n}=\frac{h}{a^{2n}}(1-n)\,r^{2n-3}.$$

Hence the force varies directly as $(2n-3)$rd power of distance from pole.

Ex. 2. *A particle describes the following curves under a force P to the pole. Find the law of force in each case.*

(i) $r^n=A\cos n\theta+B\sin n\theta.$

(ii) $r=a\sin n\theta.$

(iii) $r=a\dfrac{\cosh\theta+1}{\cosh\theta-2}$ or $r=a\dfrac{\cosh\theta-1}{\cosh\theta+2}.$

(iv) $r^2=a^2\cos^2\theta+b^2\sin^2\theta.$

(v) $\frac{1}{r^2}=a \sin^2 \theta+b \cos^2 \theta.$

(vi) $r=a \tanh\left(\frac{\theta}{\sqrt{2}}\right)$ or $r=a \coth\left(\frac{\theta}{\sqrt{2}}\right).$

Solution.

(i) $r^n=A \cos n\theta+B \sin n\theta$. Let $A=k \cos \alpha$, $B=k \sin \alpha$.

$$\therefore \quad r^n=k \cos (n\theta-\alpha).$$

Now proceeding as in Ex. 1 P. 10, we get that P varies inversely as $(2n+3)$rd power of the distance from the pole.

(ii) $r=a \sin n\theta$; $\therefore \quad u=\frac{1}{a} \operatorname{cosec} n\theta.$

$$\therefore \quad \frac{du}{d\theta}=-\frac{n}{a} \operatorname{cosec} n\theta \cot n\theta,$$

and $\frac{d^2u}{d\theta^2}=-\frac{n^2}{a} [-\operatorname{cosec} n\theta \,.\cot^2 n\theta-\operatorname{cosec}^3 n\theta]$

or $\frac{d^2u}{d\theta^2}=\frac{n^2}{a} [\operatorname{cosec} n\theta (\operatorname{cosec}^2 n\theta-1)+\operatorname{cosec}^3 n\theta]$

$$=\frac{n^2}{a} [2 \operatorname{cosec}^3 n\theta-\operatorname{cosec} n\theta]=\frac{n^2}{a} [2a^3u^3-au].$$

$$\therefore \quad P=h^2u^2\left[u+\frac{d^2u}{d\theta^2}\right]=h^2u^2 [u+n^2 . 2a^2u^3-n^2u]$$

or $P=h^2\left[\frac{2a^2n^2}{r^5}-\frac{(n^2-1)}{r^3}\right].$

(iii) From the given equation, we have

$$au=\frac{\cosh \theta-2}{\cosh \theta+1}$$

or $au=\frac{\cosh \theta+1-3}{\cosh \theta+1}=1-\frac{3}{\cosh \theta+1}.$

$$\therefore \quad a\frac{du}{d\theta}=\frac{3 \sinh \theta}{(\cosh \theta+1)^2}.$$

Differentiating again w. r. t. θ.

$$a\frac{d^2u}{d\theta^2}=3 . \frac{(\cosh \theta+1)^2 \cosh \theta-\sinh \theta . 2 (\cosh \theta+1) \sinh \theta}{(\cosh \theta+1)^4}$$

$$=3\cdot\frac{\cosh^2\theta+\cosh\theta-2(\cosh^2\theta-1)}{(\cosh\theta+1)^3}$$

$$=-3\frac{(\cosh^2\theta-\cosh\theta-2)}{(\cosh\theta+1)^3}=-3\frac{(\cosh\theta-2)}{(\cosh\theta+1)^2}.$$

$$\therefore\quad a\left(\frac{d^2u}{d\theta^2}+u\right)=\frac{\cosh\theta-2}{\cosh\theta+1}-3\frac{(\cosh\theta-2)}{(\cosh\theta+1)^2}$$

$$=\left(\frac{\cosh\theta-2}{\cosh\theta+1}\right)^2=u^2a^2.$$

Now $\quad P=h^2u^2\left(u+\dfrac{d^2u}{d\theta^2}\right)=h^2u^2\cdot\dfrac{a^2u^2}{a}=h^2a\cdot\dfrac{1}{r^4}=\dfrac{\mu}{r^4}.$

Hence the force varies inversely as fourth power of the distance from the pole.

In the second case proceed exactly as above and you will find the same answer.

(iv) The given equation is $u^{-2}=a^2\cos^2\theta+b^2\sin^2\theta$.

$$\therefore\quad -2u^{-3}\frac{du}{d\theta}=2(b^2-a^2)\sin\theta\cos\theta. \qquad \ldots(1)$$

Differentiating again w. r. t. θ, we get

$$-u^{-3}\frac{d^2u}{d\theta^2}+3u^{-4}\left(\frac{du}{d\theta}\right)^2=(b^2-a^2)(\cos^2\theta-\sin^2\theta)$$

or $$-u^{-3}\frac{d^2u}{d\theta^2}+3u^{-4}\frac{(b^2-a^2)^2\sin^2\theta\cos^2\theta}{u^{-6}}$$
$$=(b^2-a^2)(\cos^2\theta-\sin^2\theta)\quad [\text{by }(1)].$$

Multiplying both sides by $-u^3$, we get

$$\frac{d^2u}{d\theta^2}=3u^5(b^2-a^2)^2\sin^2\theta\cos^2\theta-(b^2-a^2)(\cos^2\theta-\sin^2\theta)u^3$$

$$P=h^2u^2\left[\frac{d^2u}{d\theta^2}+u\right]$$
$$=h^2u^3\,[3u^4(b^2-a^2)^2\sin^2\theta\cos^2\theta-(b^2-a^2)(\cos^2\theta-\sin^2\theta)u^2+1]$$
$$h^2u^7\,[3(b^2-a^2)^2\sin^2\theta\cos^2\theta-(b^2-a^2)(\cos^2\theta-\sin^2\theta)r^2+r^4].$$

Now putting for r^2 and simplifying, we get

$$P=h^2u^7\,[2a^4\cos^4\theta+2b^4\sin^4\theta-a^2b^2]$$

$$=h^2u^7\,[2\,(a^2+b^2)\,(a^2\cos^2\theta+b^2\sin^2\theta)-2a^2b^2\,(\cos^2\theta+\sin^2\theta)-a^2b^2]$$

$$=h^2u^7\,[2\,(a^2+b^2)\,r^2-3a^2b^2]$$

$$=h^2\,[2\,(a^2+b^2)\,r^2-3a^2b^2]\,r^{-7}.\qquad\text{Proved.}$$

(v) The given equation is $u^2=a\sin^2\theta+b\cos^2\theta$.

$$\therefore\quad u\frac{du}{d\theta}=(a-b)\sin\theta\cos\theta\qquad\ldots(1)$$

and $$\left(\frac{du}{d\theta}\right)^2+u\frac{d^2u}{d\theta^2}=(a-b)\,(\cos^2\theta-\sin^2\theta).$$

$$\therefore\quad u^2\left(\frac{du}{d\theta}\right)^2+u^3\frac{d^2u}{d\theta^2}=(a-b)\,(\cos^2\theta-\sin^2\theta)\,u^2.$$

or $$u^3\frac{d^2u}{d\theta^2}=(a-b)\,(\cos^2\theta-\sin^2\theta)\,u^2.-(a-b)^2\sin^2\theta\cos^2\theta.\text{ by (1)}$$

$$\therefore\quad u^4+u^3\frac{d^2u}{d\theta^2}=(a-b)\,(\cos^2\theta-\sin^2\theta)\,(a\sin^2\theta+b\cos^2\theta)-(a-b)^2\sin^2\theta\cos^2\theta+(a\sin^2\theta+b\cos^2\theta)^2$$

$$=a^2b^2\,(\cos^2\theta+\sin^2\theta)^2=a^2b^2\text{ (on simplification).}$$

$$\therefore\quad P=h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=h^2u^2.\frac{a^2b^2}{u^3}=\frac{h^2a^2b^2}{u}=h^2a^2b^2.r.$$

Hence P varies directly as the distance from the pole.

(vi) $r=a\tanh\dfrac{\theta}{\sqrt{2}}$.

$$\therefore\quad au=\coth\left(\frac{\theta}{\sqrt{2}}\right).$$

$$\therefore\quad a\frac{du}{d\theta}=-\frac{1}{\sqrt{2}}\operatorname{cosech}^2\frac{\theta}{\sqrt{2}}.$$

$$a\frac{d^2u}{d\theta^2}=\frac{-1}{\sqrt{2}}\cdot\frac{-2}{\sqrt{2}}\operatorname{cosech}^2\frac{\theta}{\sqrt{2}}\coth\frac{\theta}{\sqrt{2}}=\operatorname{cosech}^2\frac{\theta}{\sqrt{2}}\coth\frac{\theta}{\sqrt{2}}.$$

$$\therefore\quad au+a\frac{d^2u}{d\theta^2}=\coth\frac{\theta}{\sqrt{2}}\left(\operatorname{cosech}^2\frac{\theta}{\sqrt{2}}+1\right).$$

Now we know that $\operatorname{cosec}^2\theta-1=\cot^2\theta$ and in hyperbolic form, the corresponding formula will be

$$-\operatorname{cosech}^2\theta-1=-\coth^2\theta \text{ or } \operatorname{cosech}^2\theta+1=\coth^2\theta.$$

$$\therefore\ a\left(u+\frac{d^2u}{d\theta^2}\right)=\coth^3\frac{\theta}{\sqrt{2}}=a^3u^3.$$

$$\therefore\ P=h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=h^2u^2.a^2.u^3=h^2a^2.u^5.$$

$\therefore$ P varies as u^5 or as $\frac{1}{r^5}$ *i.e.* inversely as r^5.

Similarly proceed for $r=a\coth\frac{\theta}{\sqrt{2}}$.

Ex. 3. *A particle describes the following curves : $au=e^{n\theta}$, $n\theta$, $\cosh n\theta$ or $\sinh n\theta$ under a force P to the pole. Prove that in each case that force varies inversely as cube of the distance from the pole.*

$$a\frac{du}{d\theta}=ne^{n\theta},\ n,\ n\sinh n\theta \quad \text{or } n\cosh n\theta,$$

$$a\frac{d^2u}{d\theta^2}=n^2e^{n\theta},\ 0,\ n^2\cosh n\theta \text{ or } n^2\sinh n\theta.$$

$$\therefore\ a\left(u+\frac{d^2u}{d\theta^2}\right)=(n^2+1)\,e^{n\theta},\ n\theta,\ (n^2+1)\cosh n\theta \quad \text{or} \quad (n^2+1)\sinh n\theta$$

$$=(n^2+1)\,au,\ au,\ (n^2+1)\,au, \text{ or } (n^2+1)\,au.$$

$$\therefore\ P=h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)$$

$$=h^2u^2.(n^2+1)\ u,\ h^2u^2.u,\ h^2u^2.(n^2+1)u \text{ or } h^2u^2\,(n^2+1)\,u$$

or $P=\lambda u^3$ in each case where $\lambda=h^2\,(n^2+1)$ for all except for 2nd in which case $\lambda=h^2$.

Hence the force varies as u^3 or inversely as cube of distance from pole.

Ex. 4. *A particle describes the curve $r=ae^{\theta \cot \alpha}$ under a force P to the pole. Find the law of force. Also show that the velocity at any point is inversely proportional to the distance of the point from the centre of force.*

$$r=ae^{\theta \cot \alpha} \quad \therefore \quad 1=aue^{\theta \cot \alpha}, \text{ where } u=\frac{1}{r}$$

$$\therefore \quad \log 1=\log a+\log u+\theta \cot \alpha.$$

Differentiating, $0=\frac{1}{u}\frac{du}{d\theta}+\cot \alpha, \quad \therefore \quad \frac{du}{d\theta}=-u \cot \alpha$

and $$\frac{d^2u}{d\theta^2}=-\cot \alpha . \frac{du}{d\theta}=u \cot^2 \alpha.$$

$$\therefore \quad u+\frac{d^2u}{d\theta^2}=u+u \cot^2 \alpha=u \operatorname{cosec}^2 \alpha$$

$$\therefore \quad P=\frac{h^2}{r^2}\left(u+\frac{d^2u}{d\theta^2}\right)=\frac{h^2}{r^2}.\frac{1}{r} \operatorname{cosec}^2 \alpha=\frac{1}{r^3} h^2 \operatorname{cosec}^2 \alpha.$$

Hence the law of force is varying inversely as cube of the distance from the pole.

The pedal equation of above curve is well known to be $p=r \sin \alpha$

and $$P=\frac{h^2}{p^3}.\frac{dp}{dr}=\frac{h^2}{r^3 \sin^3 \alpha}.\sin \alpha=\frac{h^2 \operatorname{cosec}^2 \alpha}{r^3} \text{ etc.}$$

Again we know that in a central orbit $vp=h=$constant.

$\therefore \quad v.r \sin \alpha=h, \quad \because \quad p=r \sin \alpha$ is the pedal equation,

or $$v=\frac{h}{\sin \alpha}.\frac{1}{r}, \quad \textit{i. e.} \text{ velocity varies inversely as } r.$$

Converse. $\quad v=\frac{k}{r}$ given, but $v=\frac{h}{p}$.

$$\therefore \quad \frac{h}{p}=\frac{k}{r} \quad \text{or} \quad p=\frac{h}{k} r \quad \text{or} \quad p=ar.$$

Above is the pedal equation of equiangular spiral [$p=r \sin \alpha$].

Hence if the velocity varies inversely as r, then the curve is an equiangular spiral.

Cont

Ex. 5. *A particle of unit mass describes an equiangular spiral $r=ae^{\theta\cot\alpha}$ under a force which is always in a direction perpendicular to the straight line joining the particle to the pole of the spiral. Show that the force is $\mu r^{2\sec^2\alpha-3}$ and that the rate of description of the sectorial area about the pole is $\sqrt{(\mu\sin\alpha.\cos\alpha)}.r^{\sec^2\alpha}$.*

Since the force is in a direction perpendicular to the radius vector, therefore radial acceleration is zero.

$$\therefore \quad \frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=0. \qquad \ldots(1)$$

But $\quad r=ae^{\theta\cot\alpha} \quad \therefore \quad \dfrac{dr}{dt}=ae^{\theta\cot\alpha}.\cot\alpha.\dfrac{d\theta}{dt}$

or $$\frac{dr}{dt}=r\cot\alpha.\frac{d\theta}{dt}. \qquad \ldots(2)$$

$$\therefore \quad \frac{d^2r}{dt^2}-r.\frac{1}{r^2\cot^2\alpha}.\left(\frac{dr}{dt}\right)^2=0 \text{ by (1) and (2)}$$

or $$\ddot{r}-\frac{\tan^2\alpha}{r}.\dot{r}^2=0$$

Radial

Trans

or $$\frac{\ddot{r}}{\dot{r}}=\tan^2\alpha.\frac{\dot{r}}{r}$$

Integrating, we get

$$\log\dot{r}=\tan^2\alpha\log r+\log A$$

or $$\log\frac{\dot{r}}{A}=\log r^{\tan^2\alpha}. \quad \therefore \quad \dot{r}=Ar^{\tan^2\alpha}. \qquad \ldots(3)$$

Hence $r^2\dfrac{d\theta}{dt}=r.\left(r\dfrac{d\theta}{dt}\right)=r.\tan\alpha.\dfrac{dr}{dt}$ by (2)

or $r^2\dfrac{d\theta}{dt}=r\tan\alpha.Ar^{\tan^2\alpha}=A\tan\alpha.r^{1+\tan^2\alpha}=A\tan\alpha\, r^{\sec^2\alpha}$. $\qquad \ldots(4)$

Now if F be the force on a unit mass in a direction perpendicular to radius vector, then $F=\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)$

Transverse acceleration $=\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$

or $\quad F=\frac{1}{r}\frac{d}{dt}\left(A\tan\alpha\, r^{\sec^2\alpha}\right)=\frac{A\tan\alpha}{r}\,.\sec^2\alpha\,.\,r^{\sec^2\alpha-1}\,.\frac{dr}{dt}$

$$=A\tan\alpha\sec^2\alpha\,.\,r^{\sec^2\alpha-2}\,.\,Ar^{\tan^2\alpha} \text{ by (3)}$$

$$=A^2\tan\alpha\sec^2\alpha\,.\,r^{\tan^2\alpha+\sec^2\alpha-2}$$

$$=A^2\tan\alpha\sec^2\alpha\,.\,r^{2\sec^2\alpha-3}=\mu r^{2\sec^2\alpha-3}$$

where $\mu=A^2\tan\alpha\sec^2\alpha$ or $A=\sqrt{(\mu\cot\alpha\cos^2\alpha)}$.

Again rate of description of area $=\frac{1}{2}r^2\frac{d\theta}{dt}$

$$=\tfrac{1}{2}A\,.\tan\alpha\, r^{\sec^2\alpha} \text{ by (4)}$$

$$=\tfrac{1}{2}\,.\sqrt{(\mu\cot\alpha\cos^2\alpha)}\,.\tan\alpha\,.\,r^{\sec^2\alpha}$$

$$=\tfrac{1}{2}\sqrt{(\mu\sin\alpha\cos\alpha)}\,.\,r^{\sec^2\alpha}\,.$$

Ex. 6. *The velocity at any point of a central orbit is $\frac{1}{n}$th of what it would be for a circular orbit at the same distance. Show that the central force varies as $\frac{1}{r^{2n+1}}$ and that the equation of the orbit is $r^{n^2-1}=a^{n^2-1}\cos(n^2-1)\theta$.*

Let V be the velocity in a circular orbit of radius r under central force P. In the case of a circle the line joining centre to any point is along the normal.

$$\therefore\quad P=\frac{v^2}{\rho}=\frac{V^2}{r}\quad\because\ \rho=r \text{ for a circle.}\quad\therefore\quad V^2=P.r$$

$$\therefore\quad v=\frac{1}{n}V\quad\text{or}\quad v^2=\frac{1}{n^2}V^2=\frac{Pr}{n^2}=\frac{P}{n^2}\cdot\frac{1}{u}\,. \qquad \ldots(1)$$

Again we know that in a central orbit the velocity at any point is given by

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] \text{ [§ 3 Cor. P. 6]}$$

or $$\frac{P}{un^2}=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] \text{ by (1)} \qquad \ldots(2)$$

Differentiating both sides w. r. t. θ, we get

$$\frac{1}{n^2}\left[-\frac{1}{u^2}\frac{du}{d\theta}\cdot P+\frac{1}{u}\cdot\frac{dP}{du}\cdot\frac{du}{d\theta}\right]=h^2\left[2u\frac{du}{d\theta}+2\left(\frac{du}{d\theta}\right)\frac{d^2u}{d\theta^2}\right].$$

Cancel $\frac{du}{d\theta}$ from both sides and rearrange

$$-\frac{P}{u^2}+\frac{1}{u}\frac{dP}{du}=2h^2n^2\left[u+\frac{d^2u}{d\theta^2}\right].$$

But we know that $P=h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)$

$$\therefore\quad -\frac{P}{u^2}+\frac{1}{u}\frac{dP}{du}=2n^2\cdot\frac{P}{u}$$

or
$$\frac{dP}{du}=(2n^2+1)\frac{P}{u}$$

or
$$\frac{dP}{P}=(2n^2+1)\frac{du}{u}.$$

Integrating, we get

$$\log P=(2n^2+1)\log u+\log K$$

where $\log K$ is constant of integration

$$\therefore\quad \log\frac{P}{K}=\log u^{2n^2+1}\quad\text{or}\quad P=K.u^{2n^2+1}=\frac{K}{r^{2n^2+1}}.$$

Hence force varies as $\frac{1}{r^{2n^2+1}}$.

Now in order to find the equation of the path we should have a relation between r and θ *i.e.* between u and θ.

Putting for P in (2), we get

$$\frac{K.u^{2n^2+1}}{u.n^2}=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]$$

or
$$\frac{Ku^{2n^2}}{h^2n^2}-u^2=\left(\frac{du}{d\theta}\right)^2$$

or
$$\left(\frac{du}{d\theta}\right)^2=u^2\left[\frac{K}{h^2n^2}u^{2n^2-2}-1\right].$$

For the sake of convenience and judging from the form of answer given, let us put $k/h^2n^2=a^{2n^2-2}$.

$$\therefore \quad \frac{du}{d\theta}=u \cdot \sqrt{[(a^{n^2-1}\ u^{n^2-1})^2-1]}$$

or
$$\frac{du}{u\sqrt{[(a^{n^2-1}\ u^{n^2-1})^2-1]}}=\theta.$$

Put $a^{n^2-1}\ u^{n^2-1}=\sec z$.

$$\therefore \quad a^{n^2-1}\ (n^2-1)\ u^{n^2-2}\ du=\sec z \tan z\ dz.$$

$$\therefore \quad \frac{\sec z \tan z\ dz}{a^{n^2-1}.(n^2-1).u^{n^2-1}.\sqrt{(\sec^2 z-1)}}=d\theta$$

or
$$\frac{\sec z \tan z . dz}{\sec z . \tan z}=(n^2-1)\ \theta.$$

Integrating, we get
$$z=(n^2-1)\ \theta \quad \text{or} \quad \sec z=\sec (n^2-1)\ \theta$$

or
$$a^{n^2-1}\ u^{n^2-1}=\sec (n^2-1)\ \theta \quad \text{or} \quad \frac{a^{n^2-1}}{r^{n^2-1}}=\sec (n^2-1)\ \theta,$$
$$\because \quad u=1/r$$

or
$$r^{n^2-1}=a^{n^2-1} \cos (n^2-1)\ \theta.$$
Proved.

§ 5. Apses and Apsidal distance.

Definition. *An apse in a point on a central orbit at which the radius vector drawn from the centre of force is a maximum or minimum. The length of the radius vector at such a point is known as* **apsidal distance** *and the angle between two apsidal distances is called the* **apsidal angle.**

Now we know that r will be maximum or minimum according as $u\left(=\frac{1}{r}\right)$ will be minimum or maximum and from differential calculus, we know that $\frac{du}{d\theta}=0$.

Also
$$\tan \phi=r\frac{d\theta}{dr}=r.\frac{d\theta}{du}.\frac{du}{dr}$$

or $$\tan\phi = r.\frac{d\theta}{du}.\left(\frac{-1}{r^2}\right) = \frac{-1}{r}\frac{d\theta}{du} = -u\frac{d\theta}{du}.$$

$$\therefore \quad \frac{du}{d\theta} = -u \cot\phi = 0. \text{ Hence } \cot\phi = 0 \text{ or } \phi = \frac{\pi}{2}.$$

But ϕ is the angle between the tangent and radius vector. Hence we conclude that **at an apse tangent is perpendicular to radius vector.**

Properties of the apse line.

When the central acceleration is a single-valued function of the distance, then

(a) each apse-line divides the orbit symmetrically into two equal portions,

(b) there are not more than two apsidal distances (though there may be any number of apses)

§ 6. **Converse Problem.** To find the orbit when the law of force is given. We have done before that given the equation to the orbit we were to find the law of force and now when we are given the law of force we shall find the equation to the orbit. The method will be clear by the examples given below.

Ex. 1. *If a particle is projected from an apse at a distance with the velocity from infinity under the action of a central force* μr^{-2n-3}, *prove that the path is* $r^n = a^n \cos n\theta$.

$$P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = \frac{\mu}{r^{2n+3}} = \mu . u^{2n+3},$$

$$\therefore \quad h^2\left(u + \frac{d^2u}{d\theta^2}\right) = \mu u^{2n+1}.$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu . \frac{u^{2n+2}}{2n+2} + A. \qquad \ldots(1)$$

Now it is given that when $r=\infty$, *i.e.* $u=0$, $v=0$,

$\therefore \quad A=0$ and hence $v^2=\mu \cdot \dfrac{u^{2n+2}}{n+1}$.

Hence the velocity acquired at an apse where $r=a$

or $$u=\frac{1}{a} \text{ is } V^2=\frac{\mu}{(n+1)\,a^{2n+2}}. \qquad \ldots(2)$$

Now our problem is that a particle is projected from an apse with velocity V which it has acquired in falling from infinity to apse and we are to find the equation to the orbit.

Now at an apse $r=a$, *i.e.* $u=\dfrac{1}{a}$, then $v^2=V^2=\dfrac{\mu}{(n+1)\,a^{2n+2}}$ from (2).

Putting above data in (1), we get $A=0$.

Also at an apse $\dfrac{du}{d\theta}=0$.

$$\therefore \quad h^2 \cdot \frac{1}{a^2}=\frac{\mu}{(n+1)\,a^{2n+2}}, \quad \therefore \quad h^2=\frac{\mu}{(n+1)\,a^{2n}}.$$

Hence from (1), putting for h^2 and A, we get

$$\frac{\mu}{(n+1)\,a^{2n}}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\,\frac{u^{2n+2}}{n+1}$$

or $$u^2+\left(\frac{du}{d\theta}\right)^2=a^{2n} \cdot u^{2n+2}$$

or $$\frac{du}{d\theta}=u\sqrt{(a^{2n}u^{2n}-1)}$$

or $$-\frac{1}{r^2}\frac{dr}{d\theta}=\frac{1}{r}\cdot\frac{1}{r^n}\cdot\sqrt{(a^{2n}-r^{2n})}$$

or $$-\int \frac{nr^{n-1}}{\sqrt{(a^{2n}-r^{2n})}}\,dr=\int n\,d\theta$$

or $$\cos^{-1}\frac{r^n}{a^n}=n\theta+\alpha, \quad \therefore \quad r^n=a^n \cos(n\theta+\alpha).$$

Now $r=a$ when $\theta=0$; $\therefore$ $0=\alpha$.

Hence the equation to the path is $r^n=a^n \cos n\theta$.

Ex. 2. *A particle moves with a central acceleration μu^5 and is projected from an aspe distance a with velocity equal to n times that which would be acquired in falling from infinity. Show that the other apsidal distance is $\frac{a}{\sqrt{(n^2-1)}}$. If n=1 and particle is projected in any direction, show that the path is a circle passing through the centre of force.*

(Agra 1966, 34)

$$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P=\mu u^5.$$

$$\therefore \quad h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu u^3.$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\cdot\frac{u^4}{2}+A. \qquad (1)$$

Now when $r=\infty$, *i.e.* $u=0$, then $v=0$, $\therefore$ $A=0$ and therefore $v^2=\frac{\mu}{2}u^4$.

Hence the velocity at an apse is

$$V^2=\frac{\mu}{2}\cdot\frac{1}{a^4} \quad \therefore \quad V=\sqrt{\left(\frac{\mu}{2a^4}\right)}.$$

Again the particle is projected from an apse with velocity nV and we know that at an apse $u=\frac{1}{a}$, $\frac{du}{d\theta}=0$.

Hence putting in (1), we get

$$n^2V^2=\mu\cdot\frac{1}{2a^4}+A \quad \text{or} \quad n^2\frac{\mu}{2a^4}-\frac{\mu}{2a^4}=A \quad \text{or} \quad \frac{\mu}{2a^4}(n^2-1)=A$$

and $n^2V^2=h^2\left[\frac{1}{a^2}+0\right]$ or $n^2\frac{\mu}{2a^4}\cdot a^2=h^2$ or $\frac{n^2\mu}{2a^2}=h^2$.

Hence putting for A and h^2 in (1), we get

$$\frac{n^2\mu}{2a^2}\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\mu\frac{u^4}{2}+\frac{\mu}{2a^4}(n^2-1).$$

Cancel $\frac{\mu}{2}$ from both sides and multiplying by $\frac{a^2}{n^2}$, we get

$$\left(\frac{du}{d\theta}\right)^2=\frac{a^2}{n^2}\left[u^4+\frac{1}{a^4}(n^2-1)\right]-u^2$$

$$\frac{1}{n^2a^2}[a^4u^4-n^2a^2u^2+(n^2-1)]. \qquad ...(2)$$

Now we know that at an apse $\frac{du}{d\theta}=0$.

$$\therefore \quad a^4u^4-n^2a^2u^2+(n^2-1)=0. \qquad ...(3)$$

Above equation gives us the apsidal distances.

Put $u=1/r$, $\therefore$ $r^4(n^2-1)-n^2a^2r^2+a^4=0$.

Above is a quadratic in r^2 and if its roots be r_1^2 and r_2^2, then $r_1^2 . r_2^2=\frac{a^4}{n^2-1}$, $\therefore$ $r_1r_2=\frac{a^2}{\sqrt{(n^2-1)}}$.

But we are given that one apsidal distance is a, $\therefore$ $r_1=a$, hence the other apsidal distance is given by

$$a . r_2=\frac{a^2}{\sqrt{(n^2-1)}} \quad \text{or} \quad r_2=\frac{a}{\sqrt{(n^2-1)}}.$$

Again if $n=1$, then from (2), we get

$$\left(\frac{du}{d\theta}\right)^2=\frac{1}{a^2}[a^4u^4-a^2u^2]=u^2(a^2u^2-1)$$

or $$\frac{du}{d\theta}=u\sqrt{(a^2u^2-1)}$$

or $$\frac{a\, du}{au\sqrt{(a^2u^2-1)}}=d\theta,$$

Integrating $\sec^{-1} au=\theta+B$, $\because \int \frac{dx}{x\sqrt{(x^2-1)}}=\sec^{-1} x$.

When $u=\frac{1}{a}$, then $\theta=\alpha$ say, $\therefore$ $B=-\alpha$.

$\frac{1}{\sec \alpha \frac{1}{\alpha}}=\theta+B$, $\theta=\alpha$. $\because \cos 1=0$

$\cos 1=\alpha+B=0$ $\therefore B=-\alpha$.

Hence putting for A and h^2 in (1), we get

$$\frac{n^2\mu}{2a^2}\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\mu\frac{u^4}{2}+\frac{\mu}{2a^4}(n^2-1).$$

Cancel $\frac{\mu}{2}$ from both sides and multiplying by $\frac{a^2}{n^2}$, we get

$$\left(\frac{du}{d\theta}\right)^2=\frac{a^2}{n^2}\left[u^4+\frac{1}{a^4}(n^2-1)\right]-u^2$$

$$\frac{1}{n^2a^2}\left[a^4u^4-n^2a^2u^2+(n^2-1)\right]. \quad \text{...(2)}$$

Now we know that at an apse $\frac{du}{d\theta}=0$.

$$\therefore \quad a^4u^4-n^2a^2u^2+(n^2-1)=0. \quad \text{...(3)}$$

Above equation gives us the apsidal distances.

Put $u=1/r$, $\therefore$ $r^4(n^2-1)-n^2a^2r^2+a^4=0$.

Above is a quadratic in r^2 and if its roots be r_1^2 and r_2^2, then

$$r_1^2 \cdot r_2^2=\frac{a^4}{n^2-1}, \quad \therefore \quad r_1r_2=\frac{a^2}{\sqrt{(n^2-1)}}.$$

But we are given that one apsidal distance is a, $\therefore$ $r_1=a$, hence the other apsidal distance is given by

$$a \cdot r_2=\frac{a^2}{\sqrt{(n^2-1)}} \quad \text{or} \quad r_2=\frac{a}{\sqrt{(n^2-1)}}.$$

Again if $n=1$, then from (2), we get

$$\left(\frac{du}{d\theta}\right)^2=\frac{1}{a^2}\left[a^4u^4-a^2u^2\right]=u^2(a^2u^2-1)$$

or $$\frac{du}{d\theta}=u\sqrt{(a^2u^2-1)}$$

or $$\frac{a\,du}{au\sqrt{(a^2u^2-1)}}=d\theta,$$

Integrating $\sec^{-1} au=\theta+B$, $\because$ $\int\frac{dx}{x\sqrt{(x^2-1)}}=\sec^{-1}x$.

When $u=\frac{1}{a}$, then $\theta=\alpha$ say, $\therefore$ $B=-\alpha$.

∴ 1/(Sec α · 1/α) = θ + β, θ = α. ∵ Cos 1 = 0
Cos 1 = α + β = 0 ∴ β = −α.

Hence $\sec^{-1} au=\theta-\alpha$ or $au=\sec(\theta-\alpha)$

or $$\frac{a}{r}=\frac{1}{\cos(\theta-\alpha)} \quad \text{or} \quad r=a\cos(\theta-\alpha),$$

which represents a circle through the pole which is centre of force.

Ex 3. *A particle subject to a central force per unit of mass equal to* $\mu\{2(a^2+b^2)u^5-3a^2b^2u^7\}$, *is projected at a distance a with a velocity* $\frac{\sqrt{\mu}}{a}$ *in a direction at right angles to the initial distance. Show that the path is the curve*

$$r^2=a^2\cos^2\theta+b^2\sin^2\theta. \qquad \text{(Agra 1940)}$$

We know that $P=hu^2\left(u+\frac{d^2u}{d\theta^2}\right)$

$$=\mu\{2(a^2+b^2)u^5-3a^2b^2u^7\}$$

$$h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu[2(a^2+b^2)u^3-3a^2b^2u^5].$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\}=\mu[(a^2+b^2)u^4-a^2b^2u^6]+A \qquad ...(1)$$

Now we have to find the values of A and h^2 from initial conditions, when $r=a$, *i.e.* $u=\frac{1}{a}$, $\frac{du}{d\theta}=0$ at an apse, and $v^2=\frac{\mu}{a^2}$.

$$\therefore \quad \frac{\mu}{a^2}=h^2\left(\frac{1}{a^2}+0\right)=\mu\left[\frac{a^2+b^2}{a^4}-\frac{a^2b^2}{a^6}\right]+A.$$

$\therefore$ $h^2=\mu$ and $A=0$. Putting in (1), we get

$$\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu[(a^2+b^2)u^4-a^2b^2u^6]. \qquad ...(2)$$

Now $u=\frac{1}{r}$. $\quad\therefore \quad \frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$.

$$\therefore \quad \frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{a^2+b^2}{r^4}-\frac{a^2b^2}{r^6}$$

or $$\left(\frac{dr}{d\theta}\right)^2=\frac{(a^2+b^2)\,r^2-a^2b^2}{r^2}-r^2$$

or $$\left(\frac{dr}{d\theta}\right)^2=\frac{(a^2+b^2)\,r^2-a^2b^2-r^4}{r^2}=\frac{(a^2-r^2)\,(r^2-b^2)}{r^2}$$

or $$\int\frac{r\,dr}{\sqrt{\{(a^2-r^2)\,(r^2-b^2)\}}}=\int d\theta.$$

Put $r^2=a^2\cos^2 t+b^2\sin^2 t.$ **(Note the Substitution)**

$\therefore\quad 2r\,dr=2\,(b^2-a^2)\sin t\cos dt.$

Also $a^2-r^2=(a^2-b^2)\sin^2 t,\ r^2-b^2=(a^2-b^2)\cos^2 t.$

$$\therefore\quad \int\frac{(b^2-a^2)\sin t\cos t\,dt}{(a^2-b^2)\sin t\cos t}=\int d\theta.$$

or $$-t=\theta+B.$$

Initially when $\theta=0,\ r^2=a^2.$

$\therefore\quad (a^2-b^2)\sin^2 t=0$ or $t=0.$

$\therefore\quad B=0.\quad \therefore\quad t=-\theta.$

Hence $r^2=a^2\cos^2 t+b^2\sin^2 t=a^2\cos^2(-\theta)+b^2\sin^2(-\theta)$ or $r^2=a^2\cos^2\theta+b^2\sin^2\theta$ is the required equation to the path.

Alternative—If we put $a^2-r^2=t^2$ then $-2r\,dr=2t\,dt$ and $r^2-b^2=(a^2-b^2)-t^2.$

$$\therefore\quad \int\frac{t\,dt}{t\sqrt{\{(a^2-b^2)-t^2\}}}=\int d\theta \text{ or } \sin^{-1}\frac{t}{\sqrt{(a^2-b^2)}}=\theta+c.$$

When $\theta=0,\ r^2=a^2,\quad \therefore\quad t=0$ and hence $c=0.$

$\therefore\quad t^2=(a^2-b^2)\sin^2\theta$ or $a^2-r^2=a^2\sin^2\theta-b^2\sin^2\theta,$

or $\quad r^2=a^2\cos^2\theta+b^2\sin^2\theta.$

Ex. 4. *A particle moves under a force*

$$m\mu\,\{3au^4-2\,(a^2-b^2)\,u^5\},\ a>b,$$

and is projected from an apse at a distance $a+b$ *with velocity* $\sqrt{\mu}/(a+b)$. *Show that its orbit is* $r=a+b\cos\theta.$

[Agra 53, Nagpur 55 (Supp.), I. A. S. 55]

$$P=h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu\{3au^4-2(a^2-b^2)u^5\}$$

or $$h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu[3au^2-2(a^2-b^2)u^3].$$

Multiplying both sides by $2\dfrac{du}{d\theta}$ and integrating,

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu[2au^3-(a^2-b^2)u^4]+A. \quad \ldots(1)$$

We have to find the value of h^2 and A from initial conditions.

When $r=a+b$ i.e. $u=\dfrac{1}{(a+b)}$, then $\dfrac{du}{d\theta}=0$ at an apse,

and $$v^2=\frac{\mu}{(a+b)^2}\text{ given}$$

$$\therefore\quad \frac{\mu}{(a+b)^2}=h^2\left[\frac{1}{(a+b)^2}+0\right]=\mu\left[\frac{2a}{(a+b)^3}-\frac{a^2-b^2}{(a+b)^4}\right]+A.$$

$$\therefore\quad h^2=\mu \text{ and } 1=\frac{2a-(a-b)}{a+b}+A \text{ or } 1=1+A \text{ i.e. } A=0.$$

Putting for h^2 and A in (1), we get

$$\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu[2au^3-(a^2-b^2)u^4]$$

or $$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2=\frac{2a}{r^3}-\frac{(a^2-b^2)}{r^4}-\frac{1}{r^2} \qquad \because\ u=\frac{1}{r}$$

or $$\left(\frac{dr}{d\theta}\right)^2=2ar-a^2+b^2-r^2=b^2-(r-a)^2$$

or $$\frac{dr}{d\theta}=\sqrt{[b^2-(r-a)^2]} \text{ or } \frac{dr}{\sqrt{[b^2-(r-a)^2]}}=d\theta.$$

Integrating, $\sin^{-1}\dfrac{r-a}{b}=\theta+B,$

When $\theta=0,\ r=a+b$; $\therefore\ B=\sin^{-1}1=\dfrac{\pi}{2}.$

$$\therefore \quad \sin^{-1}\frac{r-a}{b}=\theta+\frac{\pi}{2} \text{ or } \frac{r-a}{b}=\sin\left(\theta+\frac{\pi}{2}\right)=\cos$$

or $r=a+b\cos\theta$ is the required equation of the orbit.

Ex. 5. *A particle moves with a central acceleration $\mu(r^5-c^4 r)$ being projected from an apse at a distance c with a velocity $c^3\sqrt{\left(\frac{2\mu}{3}\right)}$. Show that its path is $x^4+y^4=c^4$.*

(Nagpur 55 ; Rajputana 50 ; Agra 50, 54, 48, 50)

$$P=h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu\left(\frac{1}{u^5}-\frac{c^4}{u}\right), \because \quad u=\frac{1}{r}.$$

$$\therefore \quad h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu\left(\frac{1}{u^7}-\frac{c^4}{u^3}\right).$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left(-\frac{1}{3u^6}+\frac{c^4}{u^2}\right)+A. \quad \ldots(1)$$

Now we shall find the values of h^2 and A from initial conditions. We are given that $r=c$, *i.e.* $u=\frac{1}{c}$, then

$$v^2=\frac{2\mu}{3}c^6, \text{ and } \frac{du}{d\theta}=0 \text{ at an apse.}$$

$$\therefore \quad \frac{2\mu}{3}c^6=h^2\cdot\frac{1}{c^2}; \quad \therefore \quad h^2=\frac{2\mu}{3}c^8$$

and
$$\frac{2\mu}{3}c^8\left(\frac{1}{c^2}+0\right)=\mu\left(-\frac{c^6}{3}+c^6\right)+A$$

or
$$\frac{2\mu}{3}c^6=\mu\cdot\frac{2}{3}c^6+A; \quad \therefore \quad A=0.$$

Putting for h^2 and A in (1), we get

$$\frac{2\mu}{3}c^8\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left(\frac{3c^4u^4-1}{3u^6}\right)$$

or
$$\left(\frac{du}{d\theta}\right)^2=\frac{3c^4u^4-1}{2c^8u^6}-u^2=\frac{3c^4u^4+2c^8u^8-1}{2c^8u^6}$$

or $$\left(\frac{du}{d\theta}\right)^2 = -2\left[c^8u^8 - \tfrac{3}{2}c^4u^4 + \tfrac{9}{16} + \tfrac{1}{2} - \tfrac{9}{16}\right] - 2c^8u^8,$$

or $$\left(\frac{du}{d\theta}\right)^2 = \frac{\{(\frac{1}{4})^2 - (c^4u^4 - \frac{3}{4})^2\}}{c^6u^6},$$

or $$\frac{4c^4u^3\,du}{\sqrt{[(\frac{1}{4})^2 - (c^4u^4 - \frac{3}{4})^2]}} = 4\,d\theta.$$

Integrating, we get

$$\sin^{-1}\frac{(c^4u^4 - \frac{3}{4})}{\frac{1}{4}} = 4\theta + B.$$

When $r=c$, *i.e.* $u=\dfrac{1}{c}$, then $\theta=0$; $\therefore$ $\sin^{-1} 1 = 0 + B$,

i.e. $B = \dfrac{\pi}{2}$.

$$\therefore \quad 4\,(c^4u^4 - \tfrac{3}{4}) = \sin\left(4\theta + \frac{\pi}{2}\right) = \cos 4\theta$$

or $$4c^4u^4 = 3 + 2\cos^2 2\theta - 1 = 2\,(1 + \cos^2 2\theta)$$

or $$2c^4u^4 = (\cos^2\theta + \sin^2\theta)^2 + (\cos^2\theta - \sin^2\theta)^2$$

or $$2\,\frac{c^4}{r^4} = 2\,(\cos^4\theta + \sin^4\theta) \text{ or } c^4 = r^4\cos^4\theta + r^4\sin^4\theta$$

or $$x^4 + y^4 = c^4.$$ Hence proved.

Ex. 6. *A particle moves under a central* repulsive *force* $\left[\dfrac{m\mu}{(\text{distance})^3}\right]$ *and is projected from an apse at a distance a with velocity V. Show that the equation to the path is* $r\cos p\theta = a$ *and that the angle* θ *described in time t is*

$$\frac{1}{p}\tan^{-1}\left(\frac{pV}{a}t\right) \text{ where } p^2 = \frac{\mu^2 + a^2V^2}{a^2V^2}.$$

$$P = h^2u^2\left[u + \frac{d^2u}{d\theta^2}\right] = -\frac{\mu}{r^3} = -\mu u^3$$

[$-$ive sign is due to the fact that the force is a repulsive force]

or $$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = -\mu u.$$

Multiplying both sides of above by $2\,\dfrac{du}{d\theta}$ and integrating, we get

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = -\mu u^2 + A. \qquad \ldots(1)$$

Now we shall find h^2 and A from initial conditions as before.

At an apse $r = a$, or $u = \dfrac{1}{a}$ and $\dfrac{du}{d\theta} = 0$ and $v = V$

$$V^2 = h^2\left[\frac{1}{a^2} + 0\right]; \ \therefore\ h^2 = a^2V^2.$$

Also $$a^2V^2\left[\frac{1}{a^2} + 0\right] = -\mu \cdot \frac{1}{a^2} + A.$$

$\therefore\ A = \dfrac{a^2V^2 + \mu}{a^2} = p^2V^2$ by given condition.

Hence putting for h^2 and A in (1), we get

$$a^2V^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = -\mu u^2 + p^2V^2.$$

$$\therefore\ \left(\frac{du}{d\theta}\right)^2 = \frac{p^2V^2 - \mu u^2}{a^2V^2} - u^2 = \frac{p^2V^2 - u^2\,(\mu + a^2V^2)}{a^2V^2}$$

or $$\left(\frac{du}{d\theta}\right)^2 = \frac{p^2V^2 - u^2 \cdot a^2p^2V^2}{a^2V^2} = \frac{p^2}{a^2}\,(1 - a^2u^2).$$

$$\therefore\ \frac{a\,du}{\sqrt{(1 - a^2u^2)}} = p \cdot d\theta \text{ by given condition.}$$

Integrating, we get

$$\sin^{-1} au = p\theta + B.$$

When $u = \dfrac{1}{a}$, $\theta = 0$; $\therefore\ B = \dfrac{\pi}{2}$ $\quad \because\ \sin^{-1} 1 = \dfrac{\pi}{2}$.

$$\therefore\ \sin^{-1} au = p\theta + \frac{\pi}{2} \text{ or } au = \sin\left[p\theta + \frac{\pi}{2}\right]$$

or $\dfrac{a}{r} = \cos p\theta$ or $r\cos p\theta = a$ is the required equation.

Now in order to find the angle described in time t, we use the formula $h=r^2\frac{d\theta}{dt}$ or $aV=r^2\frac{d\theta}{dt}$, $\because h^2=a^2V^2$.

$$aV\,dt=a^2\sec^2 p\theta\,.\,d\theta. \qquad \because r\cos p\theta=a$$

Integrating, $Vt+C=\frac{a}{p}\tan p\theta$.

Initially, $\theta=0,\ t=0$; $\therefore C=0$.

$\therefore \frac{pV}{a}t=\tan p\theta$ or $\theta=\frac{1}{p}\tan^{-1}\left(\frac{pV}{a}t\right)$ Hence proved.

Ex. 7. (a) *A particle moves with central acceleration $\mu\left(r+\frac{2a^3}{r^2}\right)$ being projected from an apse at a distance a with twice the velocity for a circle at that distance. Find the other apsidal distance and show that the equation to the path is*

$$\frac{\theta}{2}=\tan^{-1}(t\sqrt{3})-\frac{1}{\sqrt{5}}\tan^{-1}(\sqrt{\tfrac{5}{3}}t) \text{ where } t^2=\frac{r-a}{3a-r}.$$

(Vikram 64)

In the case of a circle the central acceleration is normal acceleration which is $v^2/\rho=v^2/a$ as normal at a point always passes through the centre. If V_1 be the velocity, then by the given conditions

$$\frac{V_1^2}{a}=\mu\left(a+\frac{2a^3}{a^2}\right) \text{ where } r=a \text{ or } V_1=a\sqrt{(3\mu)}.$$

Hence the velocity of projection is $2V_1=2a\sqrt{(3\mu)}$ which is now initial velocity of projection.

$$\text{Now}\quad P=h^2u^2\left[u+\frac{d^2u}{d\theta^2}\right]=\mu\left(\frac{1}{u}+2a^3u^2\right) \qquad \because r=\frac{1}{u}$$

or

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\mu\left(\frac{1}{u^3}+2a^3\right).$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left[-\frac{1}{u^2}+4a^3u\right]+A. \qquad ..(1)$$

In order to find h^2 and A we use the initial condition that $v^2=4a^2\,(3\mu)$, $u=\frac{1}{a}$, $\frac{du}{d\theta}=0$ at an apse.

$$\therefore\quad 4a^2\,(3\mu)=h^2\left[\frac{1}{a^2}\right]=\mu\,[-a^2+4a^2]+A.$$

$\therefore\quad h^2=12\mu a^4$ and hence $12\mu a^4.\frac{1}{a^2}=\mu\,[3a^2]+A.$

$$\therefore\quad A=9\mu a^2.$$

Putting in (1), we get

$$12\mu a^4\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left[-\frac{1}{u^2}+4a^3u\right]+9\mu a^2.$$

Now put $u=\frac{1}{r}$; $\therefore\ \frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$

$$\therefore\quad 12a^4\left[\frac{1}{r^2}+\frac{1}{r^4}\left\{\frac{dr}{d\theta}\right\}^2\right]=-r^2+\frac{4a^3}{r}+9a^2$$

or
$$12a^4\left\{\frac{dr}{d\theta}\right\}^2=r^4\left[-r^2+\frac{4a^3}{r}+9a^2-\frac{12a^4}{r^2}\right]$$

$$=r^4\left[\frac{-r^4+9a^2r^2+4a^3r-12a^4}{r^2}\right]$$

$$=r^2\,(r-a)\,(-r^3-ar^2+8a^2r+12a^3)$$

$$=r^2\,(r-a)\,(r-3a)\,(-r^2-4ar-4a^2)$$

$$=r^2\,(r-a)\,(3a-r)\,(r+2a)^2.$$

$$\therefore\quad 2\sqrt{3}a^2\frac{dr}{d\theta}=r\,(r+2a)\sqrt{[(r-a)\,(3a-r)]}.$$

$$\therefore\quad \int\frac{a^2\,dr}{r\,(r+2a)\sqrt{[(r-a)\,(3a-r)]}}=\int\frac{d\theta}{2\sqrt{3}}\,. \qquad ...(2)$$

Put $r-a=(3a-r)\,t^2$ or $r\,(1+t^2)=a\,(3t^2+1)$.

$$\therefore\quad r=a\,\frac{(1+3t^2)}{1+t^2}\,;\ \therefore\ dr=a.\frac{6t\,(1+t^2)-2t\,(1+3t^2)}{(1+t^2)^2}\,dt$$

or
$$dr=\frac{4at}{(1+t^2)^2}\,dt.$$

Also $r+2a=a\cdot\dfrac{1+3t^2}{1+t^2}+2a=a\cdot\dfrac{3+5t^2}{1+t^2}$

$$3a-r=3a-a\cdot\frac{1+3t^2}{1+t^2}=a\cdot\frac{2}{(1+t^2)}.$$

Now (2) can be written as

$$\int\frac{a^2\,dr}{r\,(r+2a)\,(3a-r)\sqrt{\left\{\dfrac{r-a}{3a-r}\right\}}}=\int\frac{d\theta}{2\sqrt{3}}$$

$$=\int\frac{a^2\dfrac{4at}{(1+t^2)^2}\,dt}{\dfrac{a\,(1+3t^2)}{1+t^2}\cdot\dfrac{a\,(3+5t^2)}{(1+t^2)}\cdot\dfrac{2a}{1+t^2}\cdot t}=\frac{d\theta}{2\sqrt{3}}$$

or $$\int\frac{2\,(1+t^2)}{(1+3t^2)\,(3+5t^2)}\,dt=\frac{\theta}{2\sqrt{3}}+B.$$

Splitting into partial fractions, we get

$$\int\left[\frac{1}{1+3t^2}-\frac{1}{3+5t^2}\right]dt=\frac{\theta}{2\sqrt{3}}+B$$

or $$\frac{1}{\sqrt{3}}\tan^{-1}\sqrt{3}t-\frac{1}{\sqrt{5}\sqrt{3}}\tan^{-1}\frac{\sqrt{5}t}{\sqrt{3}}=\frac{\theta}{2\sqrt{3}}+B.$$

Initial conditions are $\theta=0$, $r=a$ and consequently $t=0$.

$$\therefore\quad B=0.$$

$$\frac{\theta}{2}=\tan^{-1}\sqrt{3}t-\frac{1}{\sqrt{5}}\tan^{-1}\frac{\sqrt{5}t}{\sqrt{3}}\text{ where }t^2=\frac{r-a}{3a-r}.$$

Again at an apse $\dfrac{du}{d\theta}=0$ or $\dfrac{-1}{r^2}\left\{\dfrac{dr}{d\theta}\right\}=0$ or $\dfrac{dr}{d\theta}=0$.

$$\therefore\quad r^2\,(r+2a)^2\,(r-a)\,(3a-r)=0$$

The positive roots of the equation are a and $3a$.

Hence the other apsidal distance is $3a$.

(b) *A particle moves with a central acceleration* $\lambda^2\,(8au^2+a^4u^5)$. *It is projected with velocity* 9λ *from an apse at a distance* $a/3$ *from origin. Show that the equation to its path is* $\dfrac{1}{\sqrt{3}}\sqrt{\left\{\dfrac{au+5}{au-3}\right\}}=\cot\dfrac{\theta}{\sqrt{6}}$.

$$P=h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=\lambda^2\,(8au^2+a^4u^5)$$

or
$$h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\lambda^2\,(8a+a^4u^3).$$

Multiplying both sides by $2\,\frac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\lambda^2\left[16au+\frac{a^4u^4}{2}\right]+A.$$

Initial conditions give

$$r=\frac{a}{3}\text{ i.e. } u=\frac{3}{a},\ v=9\lambda \text{ and } \frac{du}{d\theta}=0.$$

$$\therefore\quad 81\lambda^2=h^2\left[\frac{9}{a^2}+0\right]=\lambda^2\left[48+\frac{81}{2}\right]+A.$$

$$\therefore\quad h^2=9\lambda^2a^2 \text{ and } A=-\tfrac{15}{2}\lambda^2.$$

$$\therefore\quad 9\lambda^2a^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\lambda^2\left[16au+\frac{a^4u^4}{2}-\frac{15}{2}\right]$$

or $$9a^2\left(\frac{du}{d\theta}\right)^2=\left[\frac{a^4u^4}{2}-9a^2u^2-\frac{1}{2}\right]$$

or $$18a^2\left(\frac{du}{d\theta}\right)^2=[a^4u^4-18a^2u^2+32au-15]$$
$$=(au-1)\,[a^3u^3+a^2u^2-17au+15]$$
$$=(au-1)\,[(au-1)\,(a^2u^2+2au-15)]$$
$$=(au-1)^2\,(au+5)\,(au-3).$$

$$\therefore\quad 3\sqrt{2}a\,\frac{du}{d\theta}=(au-1)\sqrt{[(au-3)\,(au+5)]}.$$

$$\therefore\quad \frac{a\,du}{(au-3)\,(au-1)\sqrt{\dfrac{(au+5)}{(au-3)}}}=\frac{d\theta}{3\sqrt{2}}.$$

Put $au+5=(au-3)\,.\,t^2$ or $au=\dfrac{3t^2+5}{t^2-1}$.

$$\therefore\quad a\,du=\frac{(t^2-1)\,.\,6t-(3t^2+5)\,.\,2t}{(t^2-1)^2}\,dt=\frac{-16t}{(t^2-1)^2}\,dt.$$

Also $au-1=2\frac{t^2+3}{t^2-1}$ and $au-3=\frac{8}{t^2-1}$.

$$\therefore \quad \frac{-\frac{16t}{(t^2-1)^2}\,dt}{2\cdot\frac{t^2+3}{t^2-1}\cdot\frac{8}{t^2-1}\cdot t}=\frac{d\theta}{3\sqrt{2}}$$

or $$-\frac{dt}{t^2+3}=\frac{d\theta}{3\sqrt{2}}.$$

Integrating, $\frac{1}{\sqrt{3}}\cot^{-1}\frac{t}{\sqrt{3}}=\frac{\theta}{3\sqrt{2}}+B$

or $$\cot^{-1}\frac{t}{\sqrt{3}}=\frac{\theta}{\sqrt{6}}+C.$$

Initial conditions are $\theta=0$, $u=\frac{3}{a}$ *i. e.* $au=3$.

$$\therefore \quad t^2=\frac{au+5}{au-3}=\infty \text{ and } \cot^{-1}\infty=0; \quad \therefore \quad C=0.$$

Hence $$\frac{t}{\sqrt{3}}=\cot\frac{\theta}{\sqrt{6}}$$

or $$\frac{1}{\sqrt{3}}\sqrt{\left(\frac{au+5}{au-3}\right)}=\cot\frac{\theta}{\sqrt{6}}$$

(c) *A particle subject to the central acceleration* $\left(\frac{\mu}{r^3}+f\right)$ *is projected from an apse at a distance a with the velocity* $\frac{1}{a}\sqrt{\mu}$; *prove that at any subsequent time t,* $r=a-\frac{1}{2}ft^2$.

As usual, we have

$$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu u^3+f.$$

$$\therefore \quad h^2\left[u+\frac{d^2u}{d\theta^2}\right]=(\mu u+fu^{-2}).$$

Multiplying both sides by $2\dfrac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=(\mu u^2-2fu^{-1})+A \qquad ...(1)$$

When $u=\dfrac{1}{a}, \dfrac{du}{d\theta}=0$ and $v^2=\dfrac{\mu}{a^2}$.

$$\therefore \quad \frac{\mu}{a^2}=h^2 \cdot \frac{1}{a^2}=\frac{\mu}{a^2}-2fa+A.$$

$$\therefore \quad h^2=\mu \text{ and } A=2fa.$$

Putting for A and h^2 in (1), we get

$$\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=(\mu u^2-2fu^{-1})+2fa$$

or
$$\mu\left[\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right]=\mu \cdot \frac{1}{r^2}-2fr+2fa$$

or
$$\frac{\mu}{r^4} \cdot \left(\frac{dr}{dt} \cdot \frac{dt}{d\theta}\right)^2=-2fr+2fa.$$

Now $r^2\dfrac{d\theta}{dt}=h=\sqrt{\mu}; \quad \therefore \quad \dfrac{r^2}{\sqrt{\mu}}=\dfrac{dt}{d\theta}$.

Hence $\dfrac{\mu}{r^4} \cdot \dfrac{r^4}{\mu} \cdot \left(\dfrac{dr}{dt}\right)^2=-2fr+2fa.$

$$\therefore \quad \frac{dr}{dt}=\sqrt{[2f(a-r)]}.$$

$$\therefore \quad \frac{dr}{\sqrt{(a-r)}}=\sqrt{(2f)}\, dt \text{ or } -2\sqrt{(a-r)}=\sqrt{(2f)}\,.\,t+B.$$

Initially when $t=0$, $r=a$; $\therefore \quad B=0$.

Hence $4(a-r)=2f.t^2$ or $r=a-\frac{1}{2}ft^2$. **Proved.**

Ex. 8 (a) *If the law of force be* $\mu(u^4-\frac{10}{9}au^5)$ *and particle be projected from an apse at a distance 5a with a velocity equal to* $\sqrt{\frac{5}{7}}$ *of that in a circle at the same distance, show that the orbit is the limacon*

$$r=a(3+2\cos\theta). \qquad \textbf{(Agra 1964)}$$

Here we have

$$P=h^2u^2\left\{u+\frac{d^2u}{d\theta^2}\right\}=\mu\left\{u^4-\frac{10}{9}au^5\right\}$$

or

$$h^2\left\{u+\frac{d^2u}{d\theta^2}\right\}=\mu\left\{u^2-\frac{10}{9}au^3\right\}$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating

$$v^2=h^2\left[u^2+\left\{\frac{du}{d\theta}\right\}^2\right]=\mu\left[\tfrac{2}{3}u^3-\tfrac{5}{9}au^4\right]+A\,. \qquad ...(1)$$

Initial conditions give that when $r=5a$, *i. e.* $u=\frac{1}{5a}$ and V_1 be the velocity for a circle at a distance $5a$ under the same acceleration then as in Q. 7 (a) P. 34, we have

$$\frac{V_1^2}{5a}=\mu\left[\left\{\frac{1}{5a}\right\}^4-\frac{10}{9}a.\left\{\frac{1}{5a}\right\}^5\right]=\mu.\frac{7}{9}\left\{\frac{1}{5a}\right\}^4$$

$$\therefore\quad V_1=\left[\mu\cdot\frac{7}{9}\cdot\left\{\frac{1}{5a}\right\}^3\right]^{1/2}.$$

Now we are given that velocity of projection is $\sqrt{\tfrac{5}{7}}.V_1$. when $r=5a$, *i. e.* $u=\frac{1}{5a}$ and $\frac{du}{d\theta}=0.$

Hence putting in (1) we get

$$\frac{5}{7}\cdot\mu\cdot\frac{7}{9}\left\{\frac{1}{5a}\right\}^3=h^2\left[\frac{1}{(5a)^2}+0\right],$$

$$=\mu\left[\frac{2}{3}\cdot\left\{\frac{1}{5a}\right\}^3-\frac{5}{9}a\left\{\frac{1}{5a}\right\}^4\right]+A$$

$$\therefore\quad h^2=\frac{\mu}{9a}\text{ and }A=0.$$

Putting for h^2 and A in (1), we get

$$\frac{\mu}{9a}\left[u^2+\left\{\frac{du}{d\theta}\right\}^2\right]=\mu\left[\frac{2}{3}u^3-\frac{5}{9}au^4\right]$$

or $$u^2+\left\{\frac{du}{d\theta}\right\}^2=6au^3-5a^2u^4$$

or $$\left\{\frac{-1}{r^2}\frac{dr}{d\theta}\right\}^2=\frac{6a}{r^3}-\frac{5a^2}{r^4}-\frac{1}{r^2}$$

$$\therefore\quad \left\{\frac{dr}{d\theta}\right\}^2=6ar-5a^2-r^2$$

$$=4a^2-(r-3a)^2$$

$$\therefore\quad \frac{dr}{d\theta}=\pm\sqrt{[4a^2-\{r-3a\}^2]}.$$

Let us choose $-$ive sign

$$\therefore\quad \frac{dr}{-\sqrt{[4a^2-(r-3a)^2]}}=d\theta$$

or $$\cos^{-1}\frac{r-3a}{2a}=\theta+B.$$

When $r=5a$, let $\theta=0 \quad \therefore \quad B=0$

$\therefore \quad r=3a+2a\cos\theta=a\,(3+2\cos\theta)$

choosing $-$ive sign, we will have

$$\sin^{-1}\frac{r-3a}{2a}=\theta+C,$$

when $\theta=0$, $r=5a \;\therefore\; C=-\frac{\pi}{2}$

$$\therefore\quad \theta=\frac{\pi}{2}+\sin^{-1}\frac{r-3a}{2a}=\cos^{-1}\frac{r-3a}{2a}$$

or $$r=a\,(3+2\cos\theta).$$

Ex. 8 (b) *A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a. Show that the equation to its path is $r\cos(\theta/\sqrt{2})=a$.*

(Sagar B. Sc. 1965, Agra 1947)

Here we shall give only important steps.

$$\frac{V_1^2}{a}=\frac{\mu}{a^3} \quad \text{or} \quad V_1=\frac{\sqrt{\mu}}{a}.$$

Velocity of projection is $\sqrt{2}V_1=\dfrac{\sqrt{(2\mu)}}{a}$.

$$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P=\mu u^3\ ;\quad \therefore\quad h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu u.$$

Multiplying by $2\dfrac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+A.$$

Initial conditions give $h^2=2\mu$ and $A=\dfrac{\mu}{a^2}$.

$$\therefore\quad \left(\frac{du}{d\theta}\right)^2=\tfrac{1}{2}\left(\frac{1-a^2u^2}{a^2}\right) \quad \text{or} \quad \int\frac{a\,du}{\sqrt{(1-a^2u^2)}}=\int\frac{1}{\sqrt{2}}\,d\theta$$

or $\quad \sin^{-1}au=\dfrac{\theta}{\sqrt{2}}+B.$ When $\theta=0$, $u=\dfrac{1}{a}$; $\therefore\ B=\dfrac{\pi}{2}$.

$$\therefore\quad au=\sin\left(\frac{\pi}{2}+\frac{\theta}{\sqrt{2}}\right)=\cos\frac{\theta}{\sqrt{2}}$$

or $\quad a=r\cos\dfrac{\theta}{\sqrt{2}}$ is the required equation of the path.

Ex. 9 (a) *A particle moving under a constant force* from a centre *is projected in a direction perpendicular to a radius vector with velocity acquired in falling to the point of projection from the centre. Show that its path is* $\left(\dfrac{a}{r}\right)^3=\cos^2\dfrac{3\theta}{2}$ *and that the particle will ultimately move in a straight line through the origin in the same way as if the path had always been this line.*

If the velocity of projection be double that in the previous case, show that the path is

$$\frac{\theta}{2}=\tan^{-1}\sqrt{\left(\frac{r-a}{a}\right)}-\frac{1}{\sqrt{3}}\tan^{-1}\sqrt{\left(\frac{r-a}{3a}\right)}.$$

Since the force is constant say μ and is away from the centre, we have $P=-\mu$.

$$\therefore\quad h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P=-\mu$$

or

$$h^2\left(u+\frac{d^2u}{d\theta^2}\right)=-\frac{\mu}{u^2}.$$

Multiplying both sides by $2\dfrac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{2\mu}{u}+A. \qquad \text{...(1)}$$

Initially when $u=\dfrac{1}{a}$, $\dfrac{du}{d\theta}=0$ and $v^2=2\mu a$ as the velocity of projection is the velocity acquired in falling to the point of projection from the centre.

$$\therefore\quad 2\mu a=h^2\cdot\frac{1}{a^2}=2\mu a+A\ ;\quad \therefore\quad h^2=2\mu a^3 \text{ and } A=0.$$

Hence the equation (1) becomes

$$2\mu a^3\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{2\mu}{u}\quad\text{or}\quad\left(\frac{du}{d\theta}\right)^2=\frac{1}{a^3u}-u^2$$

or

$$\frac{du}{d\theta}=\frac{\sqrt{(1-a^3u^3)}}{\sqrt{(a^3u)}}$$

or

$$\int\frac{\sqrt{(a^3u)}}{\sqrt{(1-a^3u^3)}}\,du=\int d\theta$$

Put $a^{3/2}u^{3/2}=\sin z$; $\therefore$ $\frac{3}{2}a^{3/2}u^{1/2}\,du=\cos z\,dz.$

$$\therefore\quad\int\frac{2}{3}\frac{\cos z\,dz}{\sqrt{(1-\sin^2 z)}}=\theta+B\quad\text{or}\quad\frac{2}{3}z=\theta+B$$

or

$$\frac{2}{3}\sin^{-1}a^{3/2}u^{3/2}=\theta+B.$$

Initially when $\theta=0$, $u=\dfrac{1}{a}$; $\therefore$ $B=\dfrac{2}{3}\cdot\dfrac{\pi}{2}$.

$$\therefore \quad \frac{2}{3}\left[\sin^{-1} a^{3/2}u^{3/2}-\frac{\pi}{2}\right]=\theta$$

or $\quad \sin^{-1} a^{3/2}u^{3/2}=\frac{3}{2}\theta+\frac{\pi}{2} \quad$ or $\quad a^{3/2}u^{3/2}=\sin\left(\frac{3}{2}\theta+\frac{\pi}{2}\right)$

or $\quad \frac{a^{3/2}}{r^{3/2}}=\cos\frac{3\theta}{2} \quad$ or $\quad \left(\frac{a}{r}\right)^3=\cos^2\frac{3\theta}{2}$

is the required equation of the path.

When we say that particle will ultimately move, we mean that when r tends to infinity then the above equation of the path becomes $\cos^2\frac{3\theta}{2}=0$; $\therefore \frac{3\theta}{2}=\frac{\pi}{2}$ or $\theta=\frac{\pi}{3}$ which represents a straight line through the pole *i. e.* centre of force.

2nd Case. In this case the initial velocity of projection is double that in the first *i. e* initially $v=2\sqrt{(2\mu a)}$. Hence from (1), we have

$$8\mu a=h^2\cdot\frac{1}{a^2}=2\mu a+A.$$

$$\therefore \quad h^2=8\mu a^3 \text{ and } A=6\mu a.$$

Putting for h^2 and A in (1), we get

$$8\mu a^3\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{2\mu}{u}$$

Put $\quad u=\frac{1}{r}$; $\quad \therefore \quad \frac{du}{d\theta}=\frac{-1}{r^2}\frac{dr}{d\theta}.$

$$\therefore \quad 4a^3\left[\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right]=r+3a.$$

$$\therefore \quad 4a^3\left(\frac{dr}{d\theta}\right)^2=r^4$$

$=r$

$=r^2$

$=r^3$

Now put $r-a=at^2$; $\therefore$ $r=a\,(1+t^2)$
and $r+2a=a\,(3+t^2)$ and $dr=2at\,dt$.

$$4a^3 . 4a^2t^2 . \left(\frac{dt}{d\theta}\right)^2=at^2 . a^2\,(1+t^2)^2 . a^2\,(3+t^2)^2.$$

$$\therefore \quad 4\frac{dt}{d\theta}=(1+t^2)\,(3+t^2)$$

or
$$\frac{4}{(t^2+3)\,(t^2+1)}\,dt=d\theta.$$

or
$$\left(\frac{1}{t^2+1}-\frac{1}{t^2+3}\right)dt=\tfrac{1}{2}d\theta.$$

Integrating $\tan^{-1} t-\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}=\frac{\theta}{2}+B$

or
$$\tan^{-1}\sqrt{\left(\frac{r-a}{a}\right)}-\frac{1}{\sqrt{3}}\tan^{-1}\sqrt{\left(\frac{r-a}{3a}\right)}=\frac{\theta}{2}$$

$\because$ when $r=a$, $\theta=0$, $\therefore$ $B=0$.

(b) *A particle moves with a central acceleration $\mu\left(r+\frac{a^4}{r^3}\right)$ being projected from an apse at a distance a with a velocity $2\sqrt{\mu}\,.a$. Prove that it describes the curve*

$$r^2\,(2+\cos\sqrt{3}\theta)=3a^2.$$

(Agra 1955 ; Nagpur 1951, 54 ; Punjab 1952)

Here
$$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P=\mu\left(\frac{1}{u}+a^4u^3\right)$$

or
$$h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu\left(\frac{1}{u^3}+a^4u^3\right).$$

Multiplying by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left(-\frac{1}{u^2}+a^4u^2\right)+A.$$

Initial conditions give $u=\frac{1}{a}$, $\frac{du}{d\theta}=0$, $v^2=4\mu a^2$.

$\therefore$ $h^2=4\mu a^4$ and $A=4\mu a^2$ etc.

$$\therefore \quad \left(\frac{du}{d\theta}\right)^2=\frac{1}{4a^4}\left(-\frac{1}{u^2}+a^4u^2+4a^2\right)-u^2$$

$$=\frac{-1+4a^2u^2-3a^4u^4}{4a^4u^2}$$

$$=-\frac{3}{4u^2}\left(u^4-\frac{4}{3}\,\frac{u^2}{a^2}+\frac{4}{9a^4}-\frac{1}{9a^4}\right),$$

$$\frac{du}{d\theta}=\frac{\sqrt{3}}{2u}\left[\left\{\frac{1}{3a^2}\right\}^2-\left\{u^2-\frac{2}{3a^2}\right\}^2\right]^{1/2}.$$

Put $\qquad u^2-\dfrac{2}{3a^2}=t\,;\ \therefore\ 2u\,du=dt.$

$$\therefore \quad \frac{dt}{\sqrt{\left\{\left(\frac{1}{3a^2}\right)^2-t^2\right\}}}=\sqrt{3}\,d\theta.$$

Integrating, $\qquad \sin^{-1} t\,.\,3a^2=\sqrt{3}\theta+B.$

When $u=\dfrac{1}{a}$, then $t=\dfrac{1}{a^2}-\dfrac{2}{3a^2}=\dfrac{1}{3a^2}$ and $\theta=0.$

$$\therefore \quad \sin^{-1}\frac{1}{3a^2}\,.\,3a^2=0+B \quad \text{or} \quad B=\frac{\pi}{2}.$$

$$\therefore \quad t\,.\,3a^2=\sin\left\{\frac{\pi}{2}+\sqrt{3}\theta\right\}=\cos\sqrt{3}\theta$$

or
$$\left\{u^2-\frac{2}{3a^2}\right\}.\,3a^2=\cos\sqrt{3}\theta$$

or $\quad \dfrac{3a^2}{r^2}-2=\cos\sqrt{3}\theta\,;\ \therefore\ 3a^2=r^2\,(2+\cos\sqrt{3}\theta)$

is the required equation of the path.

Ex. 10. *A particle is acted on by a central* repulsive *force which varies as nth power of the distance. If the velocity at any point be equal to that which would be acquired in falling from the centre to the point, show that the equation of the path is of the form* $r^{(n+3)/2}\cos\dfrac{n+3}{2}\,\theta=\text{constant}.$

Note. In this question we are not given the initial velocity of projection and we have to find it from the given data. It is given to be same as that acquired in falling from the centre to the point, *i.e.* $r=0$ to $r=a$. We know that acceleration at any point $r=x$ is $v\frac{dv}{dx}$ and since central force P is directed towards the origin (*i.e.* x decreasing) hence we will have the equation $v\frac{dv}{dx}=-P$. Integrating the equation within proper limits of x, we shall find the value of v at any distance x.

Here in the question the force is repulsive and consequently the above equation becomes

$$v\frac{dv}{dx}=P=\mu x^n,$$

$$\int_0^V v\,dv=\int_0^a \mu x^n.$$

The limits are chosen as when $x=0$, *i.e.* at the centre, velocity is zero and when $x=a$ say, then velocity is taken to be V.

$$\therefore\ \frac{V^2}{2}=\mu.\left[\frac{x^{n+1}}{n+1}\right]_0^a\ ;\ \therefore\ V^2=2\mu\,\frac{a^{n+1}}{n+1}.$$

This value of V is the velocity of projection and rest we shall proceed as usual.

$$h^2u^2\left\{u+\frac{d^2u}{d\theta^2}\right\}=-\mu r^n=-\frac{\mu}{u^n}.$$

$$\therefore\ h^2\left\{u+\frac{d^2u}{d\theta^2}\right\}=-\frac{\mu}{u^{n+2}}.$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left\{\frac{du}{d\theta}\right\}^2\right]=\frac{2\mu}{(n+1)\,u^{n+1}}+A. \qquad ...(2)$$

For our question in which at $r=a$, velocity is V given by (1) at an apse, where $\frac{du}{d\theta}=0$. Hence from (1),

$$V^2=h^2\left(\frac{1}{a^2}+0\right)=\frac{2\mu}{(n+1)}\,a^{n+1}+A$$

or
$$\frac{2\mu}{n+1}\,a^{n+1}=\frac{h^2}{a^2}=\frac{2\mu}{n+1}\,a^{n+1}+A.$$

$$\therefore\quad A=0 \text{ and } h^2=\frac{2\mu}{n+1}\cdot a^{n+3}.$$

Hence $v^2=\frac{2\mu}{n+1}\,h^{n+3}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{2\mu}{(n+1)\,u^{n+1}}$

or $u^2+\left(\frac{du}{d\theta}\right)^2=\frac{1}{a^{n+3}\,.\,u^{n+1}}=\frac{b^2}{u^{n+1}}$ say, where $b^2=\frac{1}{a^{n+3}}$.

$$\therefore\quad \left(\frac{du}{d\theta}\right)^2=\frac{b^2}{u^{n+1}}-u^2=\frac{b^2-u^{n+3}}{u^{n+1}}.$$

$$\therefore\quad \int\frac{u^{(n+1)/2}}{\sqrt{[b^2-\{u^{(n+3)/2}\}^2]}}\,du=\int d\theta.$$

Put $u^{(n+3)/2}=t$; $\therefore\ \frac{n+3}{2}\,u^{(n+1)/2}\,du=dt.$

$$\therefore\quad \frac{2}{n+3}\int\frac{dt}{\sqrt{(b^2-t^2)}}=\int d\theta$$

or
$$\sin^{-1}\frac{t}{b}=\frac{n+3}{2}\,\theta+B.$$

Where $\theta=0$, $r=a$, *i.e.* $u=\frac{1}{a}$; $\quad\therefore\ t=\frac{1}{a^{(n+3)/2}}=b$;

$$\therefore\quad \frac{t}{b}=1 \text{ or } \sin^{-1}1=0+B \text{ or } B=\frac{\pi}{2},$$

$$\therefore\quad t=b\sin\left(\frac{\pi}{2}+\frac{n+3}{2}\,\theta\right)=\frac{1}{a^{(n+3)/2}}\cos\frac{n+3}{2}\,\theta$$

or
$$\frac{1}{r^{(n+3)/2}}=\frac{1}{a^{(n+3)/2}}\cos\frac{n+3}{2}\,\theta$$

or
$$r^{(n+3)/2}\cos\frac{n+3}{2}\,\theta=a^{(n+3)/2}.$$

Proved.

Ex. 11. (a) *A particle is projected from an apse at a distance a with the velocity from infinity, the acceleration being* μu^7. *Show that the equation to the path is* $r^2=a^2 \cos 2\theta$.

$$v\frac{dv}{dx}=-P=-\frac{\mu}{x^7}; \quad \therefore \quad \int_0^V v\,dv=\int_\infty^a -\frac{\mu}{x^7}\,dx.$$

When $x=\infty$ the velocity is 0 and when $x=a$ then velocity is say V.

$$\therefore \quad \frac{V^2}{2}=\left[\frac{\mu}{6x^6}\right]_\infty^a=\frac{\mu}{6a^6}; \quad \therefore \quad V^2=\frac{\mu}{3a^6} \qquad \ldots(1)$$

$$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P=\mu u^7 \quad \text{or} \quad h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu u^5.$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\frac{u^6}{3}+A$$

where $u=\frac{1}{a}$, $\frac{du}{d\theta}=0$, $v^2=V^2=\frac{\mu}{3a^6}$ from (1).

$$\therefore \quad h^2=\frac{\mu}{3a^4} \text{ and } A=0.$$

$$\therefore \quad \left(\frac{du}{d\theta}\right)^2=a^4u^6-u^2. \quad \text{Put } u=\frac{1}{r}; \quad \therefore \quad \frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}.$$

$$\therefore \quad \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{a^4-r^4}{r^6}; \quad \therefore \quad \frac{dr}{d\theta}=\frac{\sqrt{(a^4-r^4)}}{r}$$

or $$2r\frac{dr}{\sqrt{(a^4-r^4)}}=2\,d\theta \quad \text{or} \quad \sin^{-1}\frac{r^2}{a^2}=2\theta+B.$$

When $r=a$, $\theta=0$; $\therefore$ $B=\pi/2$.

$$\therefore \quad r^2=a^2\sin(2\theta+\pi/2) \quad \text{or} \quad r^2=a^2\cos 2\theta$$

is the required equation of the path.

Ex. 11. (b) *A particle is projected from an apse at a distance a with the velocity from infinity under the action of a central acceleration* μu^{2n+3}. *Prove that the path is*

$$r^n=a^n\cos n\theta.$$

$$v\frac{dv}{dx}=-P=-\frac{\mu}{x^{n+3}};$$

$$\therefore \quad \left[\frac{v^2}{2}\right]_0^V=\int_\infty^a -\frac{\mu}{x^{2n+3}}=\frac{\mu}{(2n+2)}\frac{1}{a^{2n+2}}.$$

$$\therefore \quad V^2=\frac{\mu}{n+1}\cdot\frac{1}{a^{2n+2}}.$$

Now $h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P=\mu u^{2n+3}$ or $h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu u^{2n+1}$.

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{\mu}{n+1}u^{2n+2}+A.$$

Initially when $u=\frac{1}{a}$, $\frac{du}{d\theta}=0$ and $v^2=V^2=\frac{\mu}{n+1}\cdot\frac{1}{a^{2n+2}}$.

$$\therefore \quad \frac{\mu}{(n+1)\,a^{2n+2}}=h^2\cdot\frac{1}{a^2}=\frac{\mu}{n+1}\cdot\frac{1}{a^{2n+2}}+A.$$

$$\therefore \quad h^2=\frac{\mu}{(n+1)\,a^{2n}} \text{ and } A=0.$$

Putting for A and h^2, we get

$$\frac{\mu}{(n+1)\,a^{2n}}\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\}=\frac{\mu u^{2n+2}}{n+1}.$$

$$\therefore \quad \left(\frac{du}{d\theta}\right)^2=u^2\,[a^{2n}.u^{2n}-1] \text{ or } \frac{du}{d\theta}=u\sqrt{(a^{2n}u^{2n}-1)}.$$

Put $u=\frac{1}{r}$; $\therefore \quad -\frac{1}{r^2}\frac{dr}{d\theta}=\frac{1}{r^{n+1}}\sqrt{(a^{2n}-r^{2n})}$.

$$\therefore \quad \frac{dr}{d\theta}=-\frac{\sqrt{(a^{2n}-r^{2n})}}{r^{n-1}}$$

or $$-\int\frac{nr^{n-1}}{\sqrt{(a^{2n}-r^{2n})}}\,dr=\int n\,d\theta \text{ or } \cos^{-1}\frac{r^n}{a^n}=n\theta+B.$$

When $\theta=0,\ r=a,\ \therefore\ B=0.$

$\therefore$ $r^n=a^n\cos n\theta$ is the required equation to the path.

Ex. 11. (c) *A particle moves in a circular orbit of radius a under the influence of an attractive force proportional to $r^{-5/2}$ directed towards the centre of the circle. When $\theta=\alpha$, its velocity is suddenly increased by a factor $\frac{2}{\sqrt{3}}$ without change of direction. Show that the polar equation of the orbit becomes* $r=a\sec^4\left(\frac{\theta-\alpha}{4}\right)$.

Here $h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P=\mu u^{5/2}$ or $h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu\,.\,u^{1/2}$.

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\,.\,\tfrac{4}{3}u^{3/2}+A. \qquad \ldots(1)$$

Now if velocity in the case of a circle be V, then

$$\frac{V^2}{a}=\mu\,.\,r^{-5/2}=\frac{\mu}{a^{5/2}}. \quad \therefore \quad V^2=\frac{\mu}{a^{3/2}}.$$

Now $v=\frac{2}{\sqrt{3}}V$ or $v^2=\frac{4}{3}V^2=\frac{4}{3}\cdot\frac{\mu}{a^{3/2}}$.

When $u=\frac{1}{a}$ and $\frac{du}{d\theta}=0$ as in the case of a circle, direction of velocity is at right angles to the radius vector.

$$\therefore \quad \frac{4}{3}\frac{\mu}{a^{3/2}}=h^2\cdot\frac{1}{a^2}=\mu\cdot\frac{4}{3}\cdot\frac{1}{a^{3/2}}+A.$$

$$\therefore \quad A=0 \text{ and } h^2=\tfrac{4}{3}\mu\sqrt{a}.$$

Hence (1) becomes

$$\frac{4}{3}\mu\sqrt{a}\left[\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right]=\mu\cdot\frac{4}{3}\cdot\frac{1}{r^{3/2}} \quad \because \quad u=\frac{1}{r}$$

or
$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{1}{r\sqrt{(ar)}}-\frac{1}{r^2}=\frac{\sqrt{r}-\sqrt{a}}{r^2\sqrt{a}}$$

or
$$\frac{\sqrt{a}}{r^2(\sqrt{r}-\sqrt{a})}\left(\frac{dr}{d\theta}\right)^2=1.$$

$$\therefore \int \frac{a^{1/4}\, dr}{r\sqrt{[\sqrt{r}-\sqrt{a}]}} = \int d\theta.$$

Put $r = a \sec^4 t$, $\therefore$ $dr = 4a \sec^4 t \tan t\, dt$.

$$\therefore \int \frac{a^{1/4}\, 4a \sec^4 t \tan t\, dt}{a \sec^4 t . a^{1/4} . \tan t} = \int d\theta$$

or $$4t = \theta + B.$$

Initially when $\theta = \alpha$, $r = a$ i.e. $\sec t = 1$, $t = 0$.

Hence $$B = -\alpha.$$

$$\therefore \quad 4t = \theta - \alpha \quad \text{or} \quad t = \frac{\theta - \alpha}{4}$$

or $$\sec^{-1}\left(\frac{r}{a}\right)^{1/4} = \frac{\theta-\alpha}{4} \quad \text{or} \quad r = a \sec^4 \frac{\theta - \alpha}{4}.$$

Ex. 12. *A particle moving with a central acceleration* $\frac{\mu}{(\text{distance})^3}$ *is projected from an apse at a distance* a *with velocity* V, *show that the path is* $r \cosh \frac{\sqrt{(\mu - a^2V^2)}}{aV}\theta = a$ *or* $r \cos\left[\frac{\sqrt{(a^2V^2-\mu)}}{aV}\theta\right] = a$ *according as* V *is* $<$ *or* $>$ *the velocity from infinity.*

V' the velocity from infinity is given by integrating

$$v\frac{dv}{dx} = -P$$

or $$\int_0^{V'} v\,dv = \int_\infty^a -\frac{\mu}{x^3}\,dx \quad \text{or} \quad \frac{V'^2}{2} = \frac{\mu}{2a^2}\ ; \ \therefore\ V' = \sqrt{\frac{\mu}{a^2}}.$$

If V is $> V'$ i.e. $V^2 > \frac{\mu}{a^2}$, then $a^2V^2 - \mu$ is $+$ive
If V is $< V'$ i.e. $V^2 < \frac{\mu}{a^2}$, then $\mu - a^2V^2$ is $+$ive } Note

where V is the velocity of projection at an apse.

$$P = h^2u^2\left[u + \frac{d^2u}{d\theta^2}\right] = \mu u^3 \quad \text{as } P = \frac{\mu}{r^3} = \mu u^3$$

or $$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\mu u.$$

Multiplying by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+A. \quad \ldots(1)$$

Initial conditions are $u=\frac{1}{a}$, $\frac{du}{d\theta}=0$, $v=V$.

$$\therefore \quad V^2=h^2\left[\frac{1}{a^2}+0\right]=\mu\cdot\frac{1}{a^2}+A.$$

$$\therefore \quad h^2=a^2V^2 \text{ and } A=\frac{a^2V^2-\mu}{a^2}.$$

Hence from (1), we get

$$\therefore \quad a^2V^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+\frac{a^2V^2-\mu}{a^2}$$

or $$a^2V^2\left(\frac{du}{d\theta}\right)^2=-(a^2V^2-\mu)\ u^2+\frac{a^2V^2-\mu}{a^2}. \quad \ldots(2)$$

Now if $a^2V^2-\mu$ is +ive, then we shall put it as

$$aV\left(\frac{du}{d\theta}\right)=\sqrt{\left(\frac{a^2V^2-\mu}{a^2}\right)}\ \sqrt{(1-a^2u^2)},$$

$$\frac{a\ du}{\sqrt{(1-a^2u^2)}}=\frac{\sqrt{(a^2V^2-\mu)}}{aV}\,d\theta \text{ Integrating}$$

or $$\sin^{-1}au=\frac{\sqrt{(a^2V^2-\mu)}}{aV}\,\theta+B.$$

When $u=1/a$, $\theta=0$; $\therefore$ $B=\pi/2$.

$$\therefore \quad au=\sin\left(\frac{\sqrt{(a^2V^2-\mu)}}{aV}\,\theta+\frac{\pi}{2}\right)=\cos\left(\frac{\sqrt{(a^2V^2-\mu)}}{aV}\,\theta\right)$$

or $$r\cos\frac{\sqrt{(a^2V^2-\mu)}}{aV}\,\theta=a \text{ is the required equation.}$$

Again if $\mu-a^2V^2$ is +ive then we shall write (2) as

$$a^2V^2\left(\frac{du}{d\theta}\right)^2=(\mu-a^2V^2)\,u^2-\frac{(\mu-a^2V^2)}{a^2}.$$

$$\left(\frac{du}{d\theta}\right)=\frac{\sqrt{(\mu-a^2V^2)}}{a}\cdot\sqrt{(a^2u^2-1)}$$

$$\frac{u}{\sqrt{(a^2u^2-1)}}=\frac{\sqrt{(\mu-a^2V^2)}}{aV}\,d\theta.$$

Integrating, we get

$$\cosh^{-1} au=\frac{\sqrt{(\mu-a^2V^2)}}{aV}\,\theta+B.$$

When $u=\frac{1}{a}$, $\theta=0$ and $\cosh^{-1} 1=0$; $\therefore$ $B=0$.

$$\therefore \quad au=\cosh\frac{\sqrt{(\mu-a^2V^2)}}{aV}\,\theta$$

or

$$r\cosh\frac{\sqrt{(\mu-a^2V^2)}}{aV}\,\theta=a.$$

Ex. 13. (a) *A particle describes an orbit with a central acceleration $\mu u^3-\lambda u^5$ being projected from an apse at a distance a with a velocity equal to that from infinity : show that its path is $r=a\cosh\frac{\theta}{n}$ where $n^2+1=\frac{2\mu a^2}{\lambda}$.*

(b) *Prove further that it will be at a distance r at the end of time*

$$\sqrt{\left(\frac{a^2}{2\lambda}\right)}\left[a^2\log\frac{r+\sqrt{(r^2-a^2)}}{a}+r\sqrt{(r^2-a^2)}\right].$$

(a) Let V be the velocity from infinity ; then

$$v\frac{dv}{dx}=-P \text{ or } \int_0^V v\,dv=-\int_\infty^a\left(\frac{\mu}{x^3}-\frac{\lambda}{x^5}\right)dx, \quad \because \quad u=\frac{1}{r}=\frac{1}{x}.$$

$$\therefore \quad \frac{V^2}{2}=\frac{\mu}{2a^2}-\frac{\lambda}{4a^4} \quad\text{or}\quad V^2=\frac{\lambda}{2a^4}\left[\frac{2\mu a^2}{\lambda}-1\right]=\frac{\lambda n^2}{2a^4}. \qquad ...(1)$$

Now proceeding as usual, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2-\lambda\frac{u^4}{2}+C. \qquad ...(2)$$

Initial conditions give

$$u=\frac{1}{a},\ v^2=V^2=\frac{\lambda}{2a^4}\,n^2 \text{ by (1) and } \frac{du}{d\theta}=0.$$

$$\therefore\ \frac{\lambda}{2a^4}\,n^2=h^2\left[\frac{1}{a^2}+0\right]=\frac{\mu}{a^2}-\frac{\lambda}{2a^4}+C=\frac{\lambda}{2a^4}\left[\frac{2\mu a^2}{\lambda}-1\right]+C$$

$$=\frac{\lambda}{2a^4}\,n^2+C.$$

$$\therefore\ h^2=\frac{\lambda}{2a^2}.n^2 \quad \text{and} \quad C=0.$$

Putting for h^2 and C in (2), we get

$$\frac{\lambda}{2a^2}\,n^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{\lambda}{2}\left[\frac{2\mu}{\lambda}\,u^2-u^4\right]$$

or
$$n^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{2\mu a^2}{\lambda}\,u^2-a^2u^4=(n^2+1)\,u^2-a^2u^4$$

or
$$n^2\left(\frac{du}{d\theta}\right)^2=u^2-a^2u^4=u^2\,(1-a^2u^2).$$

Put
$$u=\frac{1}{r} \quad \text{and} \quad \frac{du}{d\theta}=-\frac{1}{r^2}\,\frac{dr}{d\theta}.$$

$$\therefore\ n^2.\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{1}{r^2}\left(\frac{r^2-a^2}{r^2}\right).$$

$$\therefore\ \frac{dr}{\sqrt{(r^2-a^2)}}=\frac{1}{n}\,d\theta.$$

Integrating, $\cosh^{-1}\frac{r}{a}=\theta+B.$

When $\theta=0$, $r=a$ and $\cosh^{-1} 1=0$; $\therefore\ B=0.$

Hence $r=a\cosh\frac{\theta}{n}$ is the required equation to the path.

Again we know that $h=r^2\frac{d\theta}{dt}$

or
$$\sqrt{\left(\frac{\lambda}{2}\right)}.\frac{n}{a}=r^2.\frac{d\theta}{dr}.\frac{dr}{dt}=r^2.\frac{n}{\sqrt{(r^2-a^2)}}.\frac{dr}{dt}$$

$$\therefore\ \frac{dr}{dt}=\sqrt{\left(\frac{\lambda}{2a^2}\right)}\,\frac{\sqrt{(r^2-a^2)}}{r^2}$$

or $$\int \frac{r^2-a^2+a^2}{\sqrt{(r^2-a^2)}}\,dr=\int \sqrt{\left(\frac{\lambda}{2a^2}\right)}\,dt$$

or $$\int\left(\sqrt{(r^2-a^2)}+\frac{a^2}{\sqrt{(r^2-a^2)}}\right)dr=\int\sqrt{\left(\frac{\lambda}{2a^2}\right)}\,dt$$

or $$\frac{r}{a}\sqrt{(r^2-a^2)}-\frac{a^2}{2}\cosh^{-1}\frac{r}{a}+a^2\cosh^{-1}\frac{r}{a}=\sqrt{\left(\frac{\lambda}{2a^2}\right)}\,.\,t.$$

The constant of integration vanishes as when $t=0$; $r=a$.

$$\therefore\quad t=\sqrt{\left(\frac{2a^2}{\lambda}\right)}\cdot\frac{1}{2}\left[r\sqrt{(r^2-a^2)}+a^2\cosh^{-1}\frac{r}{a}\right]$$

or $$t=\sqrt{\left(\frac{a^2}{2\lambda}\right)}\left[r\sqrt{(r^2-a^2)}+a^2\log\frac{r+\sqrt{(r^2-a^2)}}{a}\right].$$

Proved.

(b) If the force at a distance r be $2m\mu\left(\frac{1}{r^3}-\frac{a^2}{r^5}\right)$ and the particle be projected from an apse at a distance a with velocity $\frac{\sqrt{\mu}}{b}$, it will be at a distance r from the centre after a time $\frac{1}{2\sqrt{\mu}}\left\{a^2\log\frac{r+\sqrt{(r^2-a^2)}}{a}+r\sqrt{(r^2-a^2)}\right\}$.

Ex. 14. *A particle moves under a central acceleration* $m\lambda(3a^3u^4+8au^2)$ *and is projected from an apse at a distance a from the centre of force with velocity* $\sqrt{(10\lambda)}$. *Show that the second apsidal distance is half of the first and that the equation to the path is*

$$2r=a\left(1+\operatorname{sech}\frac{\theta}{\sqrt{5}}\right).$$

Here $$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P=\lambda\,(3a^3u^4+8au^2)$$

or $$h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\lambda\,(3a^3u^2+8a).$$

Multiplying both sides by $2\dfrac{du}{d\theta}$ and integrating, we get

$$v^2 = h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\lambda\,(2a^2u^3+16au)+A.$$

Initial condition $u=\dfrac{1}{a},\ \dfrac{du}{d\theta}=0,\ \ v^2=10\lambda.$

$$\therefore\quad 10\lambda=h^2\left[\frac{1}{a^2}+0\right]=\lambda\,(2+16)+A.$$

$$\therefore\quad h^2=10a^2\lambda \text{ and } A=-8\lambda.$$

$$\therefore\quad 10\lambda a^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\lambda\,(2a^3u^3+16au-8)$$

or

$$5a^4\left(\frac{du}{d\theta}\right)^2=(a^3u^3-5a^2u^2+8au-4)$$

$$=(au-1)\,(a^2u^2-4au+4)$$

$$=(au-1)\,(au-2)^2.$$

$$\int\frac{a\,du}{(au-2)\sqrt{(au-1)}}=\int\frac{d\theta}{\sqrt{5}}.$$

Put $\quad au-1=t^2,\quad \therefore\ a\,du=2t\,dt.$

$$\therefore\quad \int\frac{2t\,dt}{(t^2-1)\,.\,t}=\frac{\theta}{\sqrt{5}}+B$$

or

$$2\tanh^{-1}t=\frac{\theta}{\sqrt{5}}+B.$$

When $u=\dfrac{1}{a},\ \ t=au-1=0$ and $\theta=0\,;\ \ \therefore\ B=0,$

$\because\ \tanh^{-1}0=0.$

$$\therefore\quad \sqrt{(au-1)}=\tanh\left(\frac{\theta}{2\sqrt{5}}\right).$$

Now we know from trigonometry that

$$\cos\theta=\frac{1-\tan^2\dfrac{\theta}{2}}{1+\tan^2\dfrac{\theta}{2}} \text{ and } \cosh\theta=\frac{1+\tanh^2\dfrac{\theta}{2}}{1-\tanh^2\dfrac{\theta}{2}}.$$

$$\therefore \cosh\frac{\theta}{\sqrt{5}}=\frac{1+\tanh^2\frac{\theta}{2\sqrt{5}}}{1-\tanh^2\frac{\theta}{2\sqrt{5}}}=\frac{1+(au-1)}{1-(au-1)}.$$

$$\therefore \operatorname{sech}\frac{\theta}{\sqrt{5}}=\frac{2-au}{au}=\frac{2r-a}{a}.$$

$$\therefore \quad 2r=a\left(1+\operatorname{sech}\frac{\theta}{\sqrt{5}}\right).$$

Again in order to find the second apsidal distance we know that at an apse $\frac{du}{d\theta}=0$. $\therefore$ $(au-1)(au-2)^2=0$.

$$\therefore \quad u=\frac{1}{a}, \frac{2}{a} \quad \text{or} \quad r=a \quad \text{or} \quad \frac{a}{2}.$$

Hence the other apsidal distance is $\frac{a}{2}$ which is half of the former.

Ex. 15. *A particle moves with central acceleration $\mu u^3+\lambda u^3$ and the velocity of projection at a distance R is V. Show that the particle will ultimately go off to infinity if*

$$V^2>\frac{2\mu}{R}+\frac{\lambda}{R^2}. \qquad \text{(Gauhati Hons. 65)}$$

Differential equation of the path is

$$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P=\mu u^2+\lambda u^3$$

or

$$h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu+\lambda u.$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=2\mu u+\lambda u^2+A.$$

Initially, when $u=\frac{1}{R}$, $v=V$ given.

$$h^2\left[2u\frac{du}{d\theta}+2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2}\right]=2\mu\frac{du}{d\theta}+\lambda 2u\frac{du}{d\theta}.$$

i.e. $h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=2\mu u+\lambda u^2+A.$

$$\therefore \quad V^2=\frac{2\mu}{R}+\frac{\lambda}{R^2}+A \text{ or } A=V^2-\frac{2\mu}{R}-\frac{\lambda}{R^2}.$$

$$\therefore \quad h^2\left(\frac{du}{d\theta}\right)^2=(2\mu u+\lambda u^2-u^2h^2)+A$$

or
$$h^2\left(\frac{du}{d\theta}\right)^2=u^2(\lambda-h^2)+2\mu u+A$$

or
$$=(\lambda-h^2)\left[u^2+2u\frac{\mu}{(\lambda-h^2)}+\frac{A}{\lambda-h^2}\right]$$

or
$$\frac{du}{\left[\left(u+\frac{\mu}{\lambda-h^2}\right)^2+\frac{A}{\lambda-h^2}-\frac{\mu^2}{(\lambda-h^2)^2}\right]^{1/2}}=\frac{\sqrt{(\lambda-h^2)}}{h}\,d\theta.$$

Integrate and apply

$$\int\frac{dx}{\sqrt{(x^2+a^2)}}=\log\{x+\sqrt{(x^2+a^2)}\}.$$

$$\therefore \quad \frac{\sqrt{(\lambda-h^2)}}{h}(\theta-\alpha)=\log\left[\left(u+\frac{\mu}{\lambda-h^2}\right)\right.$$
$$\left.+\sqrt{\left\{\left(u+\frac{\mu}{\lambda-h^2}\right)^2+\frac{A}{\lambda-h^2}-\frac{\mu^2}{(\lambda-h^2)^2}\right\}}\right].$$

The particle will ultimately go to infinity if a value of θ can be found for which $r=\infty$ *i.e.* $u=0$.

$$\therefore \quad \frac{\sqrt{(\lambda-h^2)}}{h}(\theta-\alpha)$$

$$=\log\left[\frac{\mu}{\lambda-h^2}+\sqrt{\left\{\left(0+\frac{\mu}{\lambda-h^2}\right)^2+\frac{A}{\lambda-h^2}-\frac{\mu^2}{(\lambda-h^2)^2}\right\}}\right].$$

$$=\log\left[\frac{\mu}{\lambda-h^2}+\frac{\sqrt{A}}{\sqrt{(\lambda-h^2)}}\right].$$

For a real value of θ to exist, A should be +ive.

Hence from (1),

$$A=V^2-\frac{2\mu}{R}-\frac{\lambda}{R^2}=+\text{ive } i.e. > 0.$$

$$\therefore \quad V^2>\frac{2\mu}{R}+\frac{\lambda}{R^2}.$$

wing relations should be remembered.

$$v^2 = h^2 \left[u^2 + \left(\frac{du}{d\theta}\right)^2 \right],$$

$$p \text{ or } \quad \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2, \quad vp = h.$$

Ex. 16. *In a central orbit the force is* $\mu u^3 (3 + 2a^2u^2)$. *If the particle be projected at a distance a with a velocity* $\sqrt{\left(\frac{5\mu}{a^2}\right)}$ *in a direction making an angle* $\tan^{-1} \frac{1}{2}$ *with the radius, show that the equation to the path is* $r = a \tan \theta$.

(Indore 66 ; Rajputana 62 ; Agra 45)

$$P = h^2u^2 \left(u + \frac{d^2u}{d\theta^2}\right) = \mu u^3 (3 + 2a^2u^2)$$

or
$$h^2 \left(u + \frac{d^2u}{d\theta^2}\right) = \mu (3u + 2a^2u^3).$$

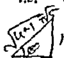
Multiplying by $2 \frac{du}{d\theta}$ and integrating, we get

$$v^2 = h^2 \left[u^2 + \left(\frac{du}{d\theta}\right)^2 \right] = \mu (3u^2 + a^2u^4) + A. \quad \ldots(1)$$

We have to use the initial conditions to find A. Here the particle is not projected from apse *i.e.* $\frac{du}{d\theta}$ is not zero in this case.

$$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{1}{p^2} \text{ and } p = r \sin \phi, \text{ where } \phi = \tan^{-1} \frac{1}{2}$$

i.e. $\tan \phi = \frac{1}{2}$ or $\sin \phi = \frac{1}{\sqrt{5}}$; $\therefore$ $p = r \sin \phi = a . \frac{1}{\sqrt{5}}$.



$$\therefore \frac{1}{p^2} = \frac{5}{a^2}.$$

Hence the initial value of $u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{1}{p^2} = \frac{5}{a^2}$ and initial velocity of projection is $\sqrt{\left(\frac{5\mu}{a^2}\right)}$ and $u = \frac{1}{a}$, $\because$ $r = a$.

Putting the above data in (1), we get

$$\frac{5\mu}{a^2}=h^2 \cdot \frac{5}{a^2}=\mu\left(\frac{3}{a^2}+\frac{1}{a^2}\right)+A.$$

$$\therefore \quad h^2=\mu \text{ and } A=\frac{\mu}{a^2}.$$

Putting the value of h^2 and A in (1), we get

$$\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left(a^2u^4+3u^2+\frac{1}{a^2}\right)$$

or
$$\left(\frac{du}{d\theta}\right)^2=\left[\frac{a^4u^4+2a^2u^2+1}{a^2}\right]=\left[\frac{a^2u^2+1}{a}\right]^2$$

or
$$\frac{a\,du}{a^2u^2+1}=d\theta.$$

Integrating, we get $\tan^{-1} au=\theta+B$.

When $\theta=0$, $r=a$, *i.e.* $u=\frac{1}{a}$.

$$\therefore \quad \frac{\pi}{4}=0+B\,; \quad \therefore \quad \tan^{-1} au=\frac{\pi}{4}+\theta$$

or
$$a=r\tan\left(\frac{\pi}{4}+\theta\right)$$

or if we turn the initial line through an angle $\frac{\pi}{4}$, the required result is obtained.

Another form. *A particle under a central acceleration $3\mu u^3+2\mu a^2u^5$ is projected from a distance a at an angle $\tan^{-1}\frac{1}{2}$ with it with a velocity equal to that in a circle at the same distance. Prove that the path is $r=a\tan\left(\frac{\pi}{4}+\theta\right)$.*

Just as in Q. 7, $\frac{V_1^2}{a}=\mu\left(\frac{3}{a^3}+2\,\frac{a^2}{a^5}\right)=\frac{5\mu}{a^3}$.

$$\therefore \quad V_1=\sqrt{\left(\frac{5\mu}{a^2}\right)}.$$

Rest is as above.

Ex. 17. *A particle of mass m moves under a central force* $\frac{m\mu}{(\text{distance})^3}$ *and is projected at a distance a from the centre of force with the velocity which at an angle α to the radius would be acquired by a fall from rest at infinity to the point of projection. Prove that the orbit is an equiangular spiral.*

$$v\frac{dv}{dx}=-P=-\frac{\mu}{x^3}\;;\quad\therefore\quad\int_0^V v\,dv=\int_\infty^a -\frac{\mu}{x^3}\,dx$$

or $$\frac{V^2}{2}=\frac{\mu}{2a^2}\;;\quad\therefore\quad V^2=\frac{\mu}{a^2}. \qquad ...(1)$$

Now proceeding as usual, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+A \qquad ...(2)$$

or $$v^2=h^2.\frac{1}{p^2}=\mu u^2+A.$$

Initially $p=r\sin\phi=a\sin\alpha$ and $v^2=\frac{\mu}{a^2}$ by (1) and $u=\frac{1}{a}$.

$$\frac{\mu}{a^2}=h^2\frac{1}{a^2\sin^2\alpha}=\frac{\mu}{a^2}+A.$$

$$\therefore\quad A=0 \text{ and } h^2=\mu\sin^2\alpha.$$

Putting in (2), we get

$$\mu\sin^2\alpha\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2$$

or $$\left(\frac{du}{d\theta}\right)^2=u^2(\operatorname{cosec}^2\alpha-1)=u^2\cot^2\alpha.$$

$$\therefore\quad\frac{du}{u}=\pm\cot\alpha.d\theta \quad\text{or}\quad \log u=\pm\theta\cot\alpha+B.$$

When $u=\frac{1}{a}$, $\theta=0$; $\therefore$ $B=\log\frac{1}{a}=-\log a$.

$\therefore$ $\log u+\log a=\pm\theta\cot\alpha$ or $\log au=\pm\theta\cot\alpha$

or $$\frac{a}{r}=e^{\pm\theta\cot\alpha} \quad\text{or}\quad r=ae^{\pm\theta\cot\alpha}.$$

Above is the well known equation of equiangular spiral.

Ex. 18. *A particle subjected to acceleration $\mu\ (u^4+2au^5)$ is projected from the point $(a, 0)$ at an angle $\cot^{-1} 2$ with the initial line with a velocity equal to the velocity from infinity. Prove that the equation to the path is*

$$r=a\ (1+2 \sin \theta). \qquad \textbf{(Agra 61)}$$

As before, $\dfrac{V^2}{2}=-\displaystyle\int_{\infty}^{a} \mu \left(\frac{1}{x^4}+\frac{2a}{x^5}\right) dx=\frac{5\mu}{6a^3}, \quad \therefore\ V^2=\frac{5\mu}{3a^3}$ ...(1)

$$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P=\mu\ (u^4+2au^5)$$

or
$$h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu\ (u^2+2au^3).$$

Multiplying both sides by $2\dfrac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left[\frac{2u^3}{3}+au^4\right]+A$$

or
$$v^2=h^2p^2=\mu\left[\frac{2u^3}{3}+au^4\right]+A.$$

Initially, $u=\dfrac{1}{a}$, $p=r \sin \phi=a \sin (\cot^{-1} 2)$

$$=a \sin\left(\sin^{-1}\frac{1}{\sqrt{5}}\right)=\frac{a}{\sqrt{5}}$$

or
$$\frac{1}{p^2}=\frac{5}{a^2} \text{ and } v^2=\frac{5\mu}{3a^3} \text{ from (1).}$$

$$\therefore\ \frac{5\mu}{3a^3}=h^2.\frac{5}{a^2}=\mu\left[\frac{2}{3a^3}+\frac{1}{a^3}\right]+A.$$

$$\therefore\ A=0 \text{ and } h^2=\frac{\mu}{3a}.$$

Putting for A and h^2 in (1), we get

$$\frac{\mu}{3a}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left[\frac{2u^3}{3}+au^4\right]$$

Initially $v^2=\frac{\mu}{2a^4}$ and $p^2=r^2\sin^2\phi=a^2\sin^2\alpha$ and $u=\frac{1}{a}$.

$$\therefore \quad \frac{\mu}{2a^4}=h^2\cdot\frac{1}{a^2\sin^2\alpha}=\frac{\mu}{2a^4}+A.$$

$$\therefore \quad A=0 \quad \text{and} \quad h^2=\frac{\mu\sin^2\alpha}{2a^2}.$$

Putting in (1), we get

$$\frac{\mu}{2a^2}\sin^2\alpha\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{\mu}{2}u^4$$

or $$\left(\frac{du}{d\theta}\right)^2=u^4a^2\operatorname{cosec}^2\alpha-u^2$$

or $$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{a^2\operatorname{cosec}^2\alpha-r^2}{r^4}.$$

$$\therefore \quad \frac{dr}{\sqrt{(a^2\operatorname{cosec}^2\alpha-r^2)}}=d\theta.$$

Integrating, $\sin^{-1}\frac{r}{a\operatorname{cosec}\alpha}=\theta+B$ when $r=a$, $\theta=0$.

$$\therefore \quad \sin^{-1}(\sin\alpha)=B \quad \text{or} \quad B=\alpha.$$

$$\therefore \quad \frac{r}{a\operatorname{cosec}\alpha}=\sin(\theta+\alpha) \quad \text{or} \quad r=a\operatorname{cosec}\alpha\sin(\theta+\alpha).$$

Above equation represents a circle whose diameter is $a\operatorname{cosec}\alpha$.

Note. $r=2a\cos(\theta-\alpha)$ or $r=2a\sin(\theta-\alpha)$ represents a circle of diameter $2a$.

Ex. 20. *A particle moves with a central acceleration $\mu\left(u^5-\frac{a^2}{8}u^7\right)$. It is projected at a distance a with a velocity $\sqrt{\frac{25}{7}}$ times the velocity for a circle at that distance at an inclination $\tan^{-1}\frac{4}{3}$ to the radius vector. Show that its path is the curve $4r^2-a^2=\frac{3a^2}{(1-\theta)^2}$.*

(Agra 52 ; I.A.S. 53 ; Mysore 66 ; Rajputana 63)

If velocity for a circle of radius a be V_1, the acceleration towards the centre *i.e.* normal acceleration is

or
$$\left(\frac{du}{d\theta}\right)^2=2au^3+3a^2u^4-u^2$$

or
$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{2ar+3a^2-r^2}{r^4}$$

or
$$\left(\frac{dr}{d\theta}\right)^2=4a^2-(r-a)^2.$$

$$\therefore \quad \frac{dr}{\sqrt{[(2a)^2-(r-a)^2]}}=d\theta.$$

Integrating, we get $\sin^{-1}\frac{r-a}{2a}=\theta+B.$

When $\theta=0,\ r=a,\ \therefore\ B=0.$

$$\therefore\ r-a=2a\sin\theta \quad \text{or} \quad r=a\,(1+2\sin\theta)$$

is the required equation of the path.

Ex. 19. *If the acceleration at a distance r is* $\frac{\mu}{r^5}$ *and the particle is projected at a distance a from the centre of force with velocity* $\sqrt{\left(\frac{\mu}{2a^4}\right)}$, *prove that the orbit is a circle through O of diameter a cosec* α, *where* α *is the inclination of the direction of projection to the radius vector.*

Differential equation to the path is

$$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P=\mu u^5$$

or
$$h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu u^3.$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\,\frac{u^4}{2}+A \qquad ...(1)$$

or
$$v^2=\frac{h^2}{p^2}=\mu\,\frac{u^4}{2}+A.$$

Initially $v^2=\frac{\mu}{2a^4}$ and $p^2=r^2\sin^2\phi=a^2\sin^2\alpha$ and $u=\frac{1}{a}$.

$$\therefore \quad \frac{\mu}{2a^4}=h^2\cdot\frac{1}{a^2\sin^2\alpha}=\frac{\mu}{2a^4}+A.$$

$$\therefore \quad A=0 \quad \text{and} \quad h^2=\frac{\mu\sin^2\alpha}{2a^2}.$$

Putting in (1), we get

$$\frac{\mu}{2a^2}\sin^2\alpha\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{\mu}{2}u^4$$

or
$$\left(\frac{du}{d\theta}\right)^2=u^4a^2\operatorname{cosec}^2\alpha-u^2$$

or
$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{a^2\operatorname{cosec}^2\alpha-r^2}{r^4}.$$

$$\therefore \quad \frac{dr}{\sqrt{(a^2\operatorname{cosec}^2\alpha-r^2)}}=d\theta.$$

Integrating, $\sin^{-1}\frac{r}{a\operatorname{cosec}\alpha}=\theta+B$ when $r=a$, $\theta=0$.

$$\therefore \quad \sin^{-1}(\sin\alpha)=B \quad \text{or} \quad B=\alpha.$$

$$\therefore \quad \frac{r}{a\operatorname{cosec}\alpha}=\sin(\theta+\alpha) \quad \text{or} \quad r=a\operatorname{cosec}\alpha\sin(\theta+\alpha).$$

Above equation represents a circle whose diameter is $a\operatorname{cosec}\alpha$.

Note. $r=2a\cos(\theta-\alpha)$ or $r=2a\sin(\theta-\alpha)$ represents a circle of diameter $2a$.

Ex. 20. *A particle moves with a central acceleration $\mu\left(u^5-\frac{a^2}{8}u^7\right)$. It is projected at a distance a with a velocity $\sqrt{\frac{25}{7}}$ times the velocity for a circle at that distance at an inclination $\tan^{-1}\frac{4}{3}$ to the radius vector. Show that its path is the curve $4r^2-a^2=\frac{3a^2}{(1-\theta)^2}$.*

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(Agra 52; I.A.S. 53; Mysore 66; Rajputana 63)

If velocity for a circle of radius a be V_1, the acceleration towards the centre *i.e.* normal acceleration is

$$\frac{V_1^2}{a}=\mu\left(\frac{1}{a^5}-\frac{a^2}{8a^7}\right)=\frac{7\mu}{8a^5},$$

$$\therefore \quad V_1^2=\frac{7\mu}{8a^4}.$$

Hence the velocity of projection is

$$V=\sqrt{\frac{25}{7}}\,V_1=\sqrt{\frac{25}{7}}\cdot\sqrt{\frac{7\mu}{8a^4}}=\sqrt{\frac{25\mu}{8a^4}}.$$

Here $\quad h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P=\mu\left(u^5-\frac{a^2}{8}u^7\right)$

or $\quad h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu\left(u^3-\frac{a^2}{8}u^5\right).$

Multiplying both sides by $2\,\frac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left(\frac{u^4}{2}-\frac{a^2u^6}{24}\right)+A. \qquad ...(1)$$

$$v^2=h^2\frac{1}{p^2}=\mu\left(\frac{u^4}{2}-\frac{a^2u^6}{24}\right)+A.$$

Initially $\quad v^2=\frac{25\mu}{8a^4},\ u=\frac{1}{a},\ p=r\sin\phi=a\cdot\frac{4}{5},\ \because\ \tan\phi=\frac{4}{3}.$

$$\therefore \quad \frac{25\mu}{8a^4}=h^2\cdot\frac{25}{16a^2}=\mu\left(\frac{1}{2a^4}-\frac{1}{24a^4}\right)+A.$$

$$\therefore \quad h^2=\frac{2\mu}{a^2}\ \text{ and }\ A=\frac{8\mu}{3a^4}.$$

Putting for h^2 and A in (1), we get

$$\frac{2\mu}{a^2}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left[\frac{u^4}{2}-\frac{a^2u^6}{24}\right]+\frac{8\mu}{3a^4}.$$

$$\therefore \quad \left(\frac{du}{d\theta}\right)^2=\frac{a^2}{2}\left[\frac{u^4}{2}-\frac{a^2u^6}{24}+\frac{8}{3a^4}\right]-u^2$$

$$=\frac{1}{48a^2}\left[64-48a^2u^2+12a^4u^4-a^6u^6\right]$$

$$=\frac{1}{48a^2}\left[4-a^2u^2\right]^3$$

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{1}{48a^2}\cdot\frac{(4r^2-a^2)^3}{r^6}$$

or $$\frac{4\sqrt{3}ar\,dr}{(4r^2-a^2)^{3/2}}=d\theta. \quad \text{Integrating,}$$

$$\frac{4\sqrt{3}a}{4.2}\times\frac{1}{-\frac{1}{2}\,(4r^2-a^2)^{1/2}}=\theta+B \quad \text{(Power formula)}$$

or $$\frac{-a\sqrt{3}}{\sqrt{(4r^2-a^2)}}=\theta+B.$$

When $\theta=0,\ r=a,\ \therefore\ B=-1.$

$$\therefore\ 1-\theta=\frac{a\sqrt{3}}{\sqrt{(4r^2-a^2)}} \quad \text{or} \quad 4r^2-a^2=\frac{3a^2}{(1-\theta)^2}$$

is the required equation of the path.

Ex. 21. (a) *A particle moves with a central acceleration $\mu\,(3u^3+a^2u^5)$ being projected from a distance a at an angle $45°$ with the distance with a velocity equal to that in a circle at the same distance. Prove that the time to the centre of force is* $\dfrac{a^2}{\sqrt{(2\mu)}}\left(2-\dfrac{\pi}{2}\right)$.

As usual, V the velocity of projection is $V^2=\dfrac{4\mu}{a^2}$

and we get $$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left(3u^2+a^2\,\frac{u^4}{2}\right)+A. \qquad \ldots(1)$$

$$p=r\sin\phi=a\sin 45°=\frac{a}{\sqrt{2}},\ v^2=4\,\frac{\mu}{a^2},\ u=\frac{1}{a}.$$

$$\therefore\ h^2=2\mu \ \text{ and } \ A=\frac{\mu}{2a^2}.$$

Putting in (1), we get

$$2\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left(3u^2+\frac{a^2u^4}{2}\right)+\frac{\mu}{2a^2}$$

or $$\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{3}{2r^2}+\frac{a^2}{4r^4}+\frac{\mu}{4a^2}$$

or $$\left(\frac{dr}{d\theta}\right)^2=\frac{1}{4a^2}\,(2a^2r^2+a^4+r^4)=\left(\frac{r^2+a^2}{2a}\right)^2.$$

Now $$\left(\frac{dr}{dt}\right)^2=\left(\frac{dr}{d\theta}\cdot\frac{d\theta}{dt}\right)^2.$$

But $r^2 \dfrac{d\theta}{dt}=h=\sqrt{(2\mu)}$; $\therefore$ $\dfrac{d\theta}{dt}=\dfrac{\sqrt{(2\mu)}}{r^2}$.

$\therefore$ $\left(\dfrac{dr}{dt}\right)^2=\left\{\dfrac{r^2+a^2}{2a}\cdot\dfrac{\sqrt{(2\mu)}}{r^2}\right\}^2$ or $\dfrac{dr}{dt}=-\dfrac{r^2+a^2}{2a}\,\dfrac{\sqrt{(2\mu)}}{r^2}$

as $\dfrac{dr}{dt}$ stands for rate of increase of r w. r. t. t. Here r is decreasing when the particle reaches the centre.

or
$$-\frac{r^2}{r^2+a^2}\,dr=\frac{\sqrt{(2\mu)}}{2a}\,dt$$

or
$$-\int_a^0 \left(1-\frac{a^2}{r^2+a^2}\right) dr=\int_{t=0}^{T} \frac{\sqrt{(2\mu)}}{2a}\,dt$$

or
$$\left[r-\frac{1}{a}\cdot a^2 \tan^{-1}\frac{r}{a}\right]_0^a=\frac{\sqrt{(2\mu)}}{2a}\,T$$

or
$$\left(a-a\,\frac{\pi}{4}\right)=\frac{\sqrt{(2\mu)}}{2a}\,T$$

or
$$T=\frac{a^2}{\sqrt{(2\mu)}}\left(2-\frac{\pi}{2}\right).$$

Ex. 21. (b) *A particle of mass m moves under a central attractive force* $m\mu\left(\dfrac{5}{r^3}+\dfrac{8c^2}{r^5}\right)$ *and is projected from an apse at a distance* c *with velocity* $\dfrac{3\sqrt{\mu}}{c}$, *prove that the orbit is* $r=c\cos\frac{2}{3}\theta$ *and that it will arrive at the origin after a time* $\dfrac{\pi c^2}{8\sqrt{\mu}}$. **(Calcutta Hon. 1963 ; Rajputana M. Sc. '65)**

$$P=h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu\,(5u^3+8c^2u^5)$$

or
$$h^2\left(u+\frac{d^2u}{d\theta}\right)=\mu\,(5u+8c^2u^3)$$

Multiplying both sides by $2\,\dfrac{du}{d\theta}$ and integrating

$$v^2=h^2\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\}=\mu\,(5u^2+4c^2u^4)+A \qquad \ldots(1)$$

Initially when $r=c$ *i.e.* $u=\frac{1}{c}$, $v^2=\frac{9\mu}{c^2}$ and $\frac{du}{d\theta}=0$ at an apse.

$$\therefore \quad \frac{9\mu}{c^2}=h^2 \cdot \frac{1}{c^2}=\mu\left(\frac{5}{c^2}+\frac{4}{c^2}\right)+A.$$

$$\therefore \quad h^2=9\mu \text{ and } A=0.$$

Putting in (1), we get

$$9\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\,(5u^2+4c^2u^4)$$

or
$$9\left(\frac{du}{d\theta}\right)^2=4c^2u^4-4u^2=4u^2\,(c^2u^2-1).$$

$$\therefore \quad \int \frac{c\,du}{cu\sqrt{(c^2u^2-1)}}=\int \tfrac{2}{3}\,d\theta.$$

Integrating, $\sec^{-1} cu=\frac{2}{3}\,\theta+B.$

When $u=\frac{1}{c}$, $\theta=0 \quad \therefore \quad B=0,$

$$\therefore \quad \sec^{-1} cu=\tfrac{2}{3}\theta \text{ or } \frac{c}{r}=\sec \tfrac{2}{3}\theta$$

or
$$r=c\cos\tfrac{2}{3}\theta. \qquad \text{...(2)}$$

Time to reach the origin. We know that $r^2\frac{d\theta}{dt}=h.$

or
$$\int_0^{3\pi/4} c^2\cos^2\tfrac{2}{3}\,\theta\,d\theta=\int_0^t h\,dt.$$

When $t=0$, $r=c$ and from (2) $\theta=0$,

When $t=t$, $r=0$ (at origin) and from (2), $\theta=\frac{3\pi}{4}.$

Put $\frac{2}{3}\,\theta=x$, $\therefore$ $d\theta=\frac{3}{2}\,dx$ and limits for x become 0 to $\pi/2$.

$$\therefore \quad \int_0^{\pi/2} c^2\cos^2 x\,.\,\tfrac{3}{2}\,dx=ht$$

or
$$\tfrac{3}{2}c^2\,.\,\tfrac{1}{2}\,.\,\pi/2=\sqrt{(9\mu)}\,.\,t,$$

$$\therefore \quad t=\frac{\pi c^2}{8\sqrt{\mu}}, \quad \because \quad h^2=9\mu.$$

Ex. 22. (a) *A particle is projected with velocity* $\sqrt{\left(\frac{2\mu}{3c^3}\right)}$ *from a point P in a field of attractive force* $\frac{\mu}{r^4}$ *to a point O distant c from P where r denotes the distance from O. If the dirction of projection makes an angle 45° with PO, prove that the orbit is a cardioid and the particle will arrive at O after a time* $\left(\frac{3\pi}{4}-2\right)\sqrt{\left(\frac{3c^5}{\mu}\right)}$.

Proceeding as usual we shall get

$$\frac{\mu}{3c}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{2\mu}{3}u^3$$

$$\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{2c}{r^3}.$$

$$\therefore \quad \left(\frac{dr}{d\theta}\right)^2=2cr-r^2=c^2-(r-c)^2.$$

$$\therefore \quad \frac{dr}{\sqrt{[c^2-(r-c)^2]}}=d\theta \quad \text{or} \quad \sin^{-1}\frac{r-c}{c}=\theta+B$$

$\theta=0$ when $r=c$; $\therefore$ $B=0$.

$\therefore$ $r=c\,(1+\sin\theta)$ which is the equation of a cardioid.

Again, $r^2\frac{d\theta}{dt}=h$ or $\int_0^{-\pi/2} c^2\,(1+\sin\theta)^2\,d\theta$

$$=\int_0^T \sqrt{\left(\frac{\mu}{3c}\right)}\,dt.$$

When $t=0$, $r=c$; $\therefore$ $\theta=0$, when $t=T$, $r=0$.

$$\therefore \quad \sin\theta=-1, \quad i.e. \quad \theta=-\frac{\pi}{2}$$

$$\int_0^{-\pi/2} c^2\left(1+2\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta=\sqrt{\left(\frac{\mu}{3c}\right)}\,T$$

$$\frac{c^2}{2}\left[3\theta-4\cos\theta+\frac{\sin 2\theta}{2}\right]_0^{-\pi/2}=\sqrt{\left(\frac{\mu}{3c}\right)}\,T$$

$$\frac{c^2}{2}\left[-\frac{3\pi}{2}-4\,(0-1)\right]=\sqrt{\left(\frac{\mu}{3c}\right)}T.$$

$$\therefore \quad T=-\sqrt{\left(\frac{3c^5}{\mu}\right)}\left[\frac{3\pi}{4}-2\right]=\sqrt{\left(\frac{3c^5}{\mu}\right)}\left[\frac{3\pi}{4}-2\right]$$

rejecting —ive sign.

Ex. 22. (b) *A particle under a central acceleration* $\frac{\mu}{r^7}$ *is projected from a point at a distance a with a velocity from infinity at an angle* α *with the initial line. Prove that the time of arriving at the centre of force is* $\frac{1}{4}\left(\frac{a}{V}\right)\sec^2\frac{\alpha}{2}$, *where V is the velocity of projection.*

Proceeding as in Q. 11, velocity of projection V is given by $\frac{\mu}{3a^6}$ and we have after integration the equation

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\frac{u^6}{3}+A \qquad \ldots(1)$$

or

$$v^2=h^2 \cdot \frac{1}{p^2}=\frac{\mu u^6}{3}+A.$$

Initially $v^2=V^2=\frac{\mu}{3a^6}$, $u=\frac{1}{a}$, $p^2=r^2\sin^2\phi=a^2\sin^2\alpha$.

$$\therefore \quad \frac{\mu}{3a^6}=h^2 \cdot \frac{1}{a^2\sin^2\alpha}=\frac{\mu}{3a^6}+A\,; \quad \therefore \quad A=0$$

and

$$h^2=\frac{\mu\sin^2\alpha}{3a^4}.$$

Putting for h^2 and A in (1), we get

$$\frac{\mu\sin^2\alpha}{3a^4}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{\mu}{3}u^6$$

or

$$\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{a^4}{r^6\sin^2\alpha}.$$

or

$$\left(\frac{dr}{d\theta}\right)^2=\frac{a^4-r^4\sin^2\alpha}{r^2\sin^2\alpha}.$$

Now $r^2\frac{d\theta}{dt}=h=\sqrt{\left(\frac{\mu}{3}\right)}\frac{\sin\alpha}{a^2}$; $\therefore \quad \frac{d\theta}{dt}=\sqrt{\left(\frac{\mu}{3}\right)}\frac{\sin\alpha}{a^2r^2}$.

$$\therefore \quad \left(\frac{dr}{dt}\right)^2=\left(\frac{dr}{d\theta}\cdot\frac{d\theta}{dt}\right)^2=\frac{a^4-r^4\sin^2\alpha}{r^2\sin^2\alpha}\cdot\frac{\mu}{3}\cdot\frac{\sin^2\alpha}{a^4r^4}.$$

$$\therefore \quad \frac{dr}{dt} = -\sqrt{\left(\frac{\mu}{3}\right)} \frac{\sqrt{(a^4 - r^4 \sin^2 \alpha)}}{a^2 r^3}$$

as r decreases with the increase of t and hence $-$ive sign

or $$-\int_a^0 \frac{r^3}{\sqrt{(a^4 - r^4 \sin^2 \alpha)}}\, dr = \int_{t=0}^{T} \sqrt{\left(\frac{\mu}{3}\right)} \cdot \frac{1}{a^2}\, dt$$

or $$\frac{1}{4 \sin^2 \alpha} \cdot \left[2\sqrt{(a^4 - r^4 \sin^2 \alpha)} \right]_a^0 = \sqrt{\left(\frac{\mu}{3}\right)} \cdot \frac{1}{a^2} \cdot T$$

or $$\frac{1}{2 \sin^2 \alpha} \left[a^2 - a^2 \cos \alpha\right] = \sqrt{\left(\frac{\mu}{3}\right)} \cdot \frac{1}{a^2}\, T$$

or $$\frac{1}{2\,(1+\cos \alpha)} = \sqrt{\left(\frac{\mu}{3}\right)} \cdot \frac{1}{a^4} \cdot T$$

or $$\frac{1}{4 \cos^2 \dfrac{\alpha}{2}} = \frac{Va^3}{a^4} \cdot T \quad \because \quad V^2 = \frac{\mu}{3a^6}.$$

$$\therefore \quad T = \frac{1}{4} \left(\frac{a}{V}\right) \sec^2 \frac{\alpha}{2}.$$

Ex. 22. (c) *A particle under a central acceleration μr^3 is projected at a distance a with a velocity from infinity. Prove that the orbit is an equiangular spiral and the time to reach a distance r from the centre is $\frac{1}{2}\,(r^2 - a^2)\, \dfrac{\sec \beta}{\sqrt{\mu}}$ where β is the angle in the spiral between the tangent and radius-vector.*

(Ref. Q. 17)

Ex. 22. (d) *Show that a particle can describe a rectangular hyperbola under a force from a fixed centre varying as the distance and show that the time the radius vector to the particle from the centre takes in sweeping out an angle θ from vertex is given by $\tan \theta = \tanh (t\sqrt{\mu})$, μ being acceleration at unit distance.*

The equation of rectangular hyperbola is $x^2 - y^2 = a^2$ with vertex at $(a, 0)$ or in polar form its equation is

$$r^2 (\cos^2 \theta - \sin^2 \theta) = a^2 \quad \text{or} \quad r^2 \cos 2\theta = a^2.$$

We will find its pedal equation. Taking log, we get

$$2 \log r + \log \cos 2\theta = \log a^2. \qquad \text{Differentiating.}$$

$$\therefore \quad 2 \cdot \frac{1}{r}\frac{dr}{d\theta}+\frac{1}{\cos 2\theta}(-2\sin 2\theta)=0$$

or

$$\cot\phi=\tan 2\theta \; ; \quad \therefore \quad \phi=90-2\theta$$

$$p=r\sin\phi=r\cos 2\theta=r\frac{a^2}{r^2}$$

$$\therefore \quad p=\frac{a^2}{r} \text{ and } \frac{dp}{dr}=-\frac{a^2}{r^2}. \qquad \ldots(1)$$

Since the force is from the centre (not towards the centre)

$$\therefore \quad P=-\mu r.$$

Also we know that $P=\dfrac{h^2}{p^3}\dfrac{dp}{dr}$.

$$\therefore \quad -\mu r=\frac{h^2}{p^3}\cdot\left(-\frac{a^2}{r^3}\right). \qquad \ldots(2)$$

Initially when at the vertex $r=a$ $\quad\therefore\ p=a$ from (1).

$\therefore$ from (2), we get $h^2=\mu a^4$.

Now $h=r^2\dfrac{d\theta}{dt}$ or $\sqrt{\mu}\,.a^2.dt=r^2\,d\theta$

or

$$\int_{t=0}^{t}\sqrt{\mu}a^2\,dt=\int\frac{a^2}{\cos 2\theta}\,d\theta=\int_{\theta=0}^{\theta}a^2\sec 2\theta\,.d\theta$$

$$\therefore \quad \sqrt{\mu}a^2t=\tfrac{1}{2}a^2\log\tan\left(\frac{\pi}{4}+\theta\right)$$

$$\int\sec\theta\,d\theta=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

Also at the vertex $\theta=0$.

$$\therefore \quad \sqrt{\mu}t=\tfrac{1}{2}\log\frac{1+\tan\theta}{1-\tan\theta}=\tanh^{-1}\tan\theta$$

$$\therefore \quad \tanh\sqrt{\mu}t=\tan\theta. \qquad \text{Hence proved.}$$

$$\because \quad \frac{1}{2}\log\frac{1+x}{1-x}=\tanh^{-1}x.$$

Ex. 22. (c) *Show that if the law of force is $\mu(r-a)^{-2}$ and the particle is projected with a velocity from infinity from a point at a distance c, $(2a>c>a)$, at an angle $2\cos^{-1}\sqrt{(a/c)}$, it will describe the orbit.*

$$\frac{\theta}{2}=\tanh^{-1}\sqrt{\left(\frac{r-a}{a}\right)}-\tan^{-1}\sqrt{\left(\frac{r-a}{a}\right)},$$

we have $h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P=\frac{\mu}{(r-a)^2}=\mu\frac{u^2}{(1-au)^2}$

or
$$h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\frac{\mu}{(1-au)^2}.$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{2\mu}{a(1-au)}+A$$

or
$$v^2=h^2\frac{1}{p^2}=\frac{2\mu}{a(1-au)}+A. \quad \ldots(1)$$

Initially, $\phi=2\cos^{-1}\sqrt{\left(\frac{a}{c}\right)}$ *i.e.* $\cos\frac{\phi}{2}=\sqrt{\left(\frac{a}{c}\right)}$.

$$\therefore \sin\frac{\phi}{2}=\sqrt{\left(1-\frac{a}{c}\right)}=\sqrt{\left(\frac{c-a}{c}\right)}.$$

$$\therefore \sin\phi=2\sin\frac{\phi}{2}\cos\frac{\phi}{2}=\frac{2}{c}\sqrt{\{a(c-a)\}}.$$

Also if V be the velocity of projection, then as in Q. 17.

$$\int_0^V v\,dv=-\int_\infty^c\frac{\mu}{(r-a)^2}\,dr \text{ or } \frac{V^2}{2}=\frac{\mu}{c-a}. \quad \therefore V^2=\frac{2\mu}{c-a}.$$

$\therefore$ Initially $p=r\sin\phi=c\cdot\frac{2}{c}\sqrt{\{a(c-a)\}}$

and
$$v^2=\frac{2\mu}{c-a},\ u=\frac{1}{c}.$$

$$\therefore \frac{2\mu}{c-a}=h^2\cdot\frac{1}{4a(c-a)}=\frac{2\mu\cdot c}{a(c-a)}+A.$$

$$\therefore h^2=8a\mu \text{ and } A=\frac{2\mu}{c-a}\left[1-\frac{c}{a}\right]=\frac{-2\mu}{a}.$$

Putting for h^2 and A in (1), we get

$$8a\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{2\mu}{a(1-au)}-\frac{2\mu}{a}=\frac{2\mu}{a}\cdot\frac{au}{(1-au)}$$

or $$\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{1}{4a}\cdot\frac{1}{r-a}$$

or $$\left(\frac{dr}{d\theta}\right)^2=\frac{r^3}{4a\,(r-a)}-r^2=\frac{r^2[r^2-4ar+4a^2]}{4a\,(r-a)}$$

or $$\frac{dr}{d\theta}=\frac{r\,.(2a-r)}{2\sqrt{(a)}\sqrt{(r-a)}}$$

$$\int\frac{\sqrt{(a)}\sqrt{(r-a)}}{r\,.[a-(r-a)]}\,dr=\int\frac{d\theta}{2}.$$

Put $r-a=at^2$; $\therefore$ $dr=2at\ dt.$

$$\int\frac{\sqrt{(a)}\,.\sqrt{a}\,.\,t}{a\,(1+t^2)\,a\,(1-t^2)}\,.2at\ dt=\frac{\theta}{2}$$

$$\int\left(\frac{1}{1-t^2}-\frac{1}{1+t^2}\right)\ dt=\frac{\theta}{2}$$

or $\tanh^{-1} t-\tan^{-1} t=\frac{\theta}{2}$ where $t=\sqrt{\left(\frac{r-a}{a}\right)}$.

Ex. 22. (f) *A particle is attracted towards a centre of force S with a force $\lambda\ (r^{-2}+2cr^{-3})$ per unit of mass, where r denotes the distance from S. It is projected from a point P at a distance c from S with a velocity $\frac{4}{3}\sqrt{\left(\frac{2\lambda}{c}\right)}$ at an angle $\frac{\pi}{3}$ with PS. Prove that*

$$\frac{d^2u}{d\theta^2}+\frac{1}{4}\,u=\frac{3}{8c}$$

where θ is the angle between SP and the radius vector to the particle and $u=1/r$. Find also the polar equation of the orbit.

Ex. 22. (g) *A particle is projected with velocity $\sqrt{\left(\frac{2\mu}{3a^3}\right)}$ at right angles to the radius vector at a distance a from the centre of attracting force μ/r^4 per unit of mass. Find the path of the particle and show that the time it takes to reach the centre of force is $\frac{3}{8}\pi\sqrt{\left(\frac{3a^5}{2\mu}\right)}$.*

Ex. 22. (h) *A particle moves in a plane under an attraction μ/r^4 per unit mass towards a centre of attraction O in the plane. It is projected from a point A at distance a from O with velocity u at right angles to OA. Prove that the differential equation of the orbit is*

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{a^2u^2}(r^2-a^2)\left\{\left(u^2-\frac{\mu}{2a^3}\right)r^2-\frac{\mu}{2a^2}\right\}.$$

Prove that in a special case when $u^2=\mu/2a^3$ the orbit is the circle on OA as diameter and that the time from A to O is $\pi a^3/\sqrt{(8\mu)}$

Ex. 23. *A particle moves in a curve under a central acceleration so that its velocity at any point is equal to that in a circle at the same distance and under the same attraction. Show that the law of force is that of inverse cube and that the path is an equiangular spiral.* (Agra 46 ; Rajputana 53)

We know that $vp=h$ and $\frac{h^2}{p^3}\frac{dp}{dr}=P$.

If v be the velocity at any point r, then for the case of a circle the velocity is given to be the same at the same distance and under the same attraction.

$$\therefore \quad P=\frac{v^2}{r} \text{ but } vp=h \; ; \quad \therefore \quad P=\frac{h^2}{p^2r}. \qquad ...(1)$$

Now we know that $\frac{h^2}{p^3}\frac{dp}{dr}=P=\frac{h^2}{p^2r}$ by (1)

or $\quad \frac{1}{p}\,dp=\frac{1}{r}\,dr \quad$ or $\quad \log p=\log r+\log A=\log rA.$

$\therefore \; p=Ar$ *i.e.* p varies as r. Now the only curve in which p varies as r is equiangular spiral [$r=ae^{\theta \cot \alpha}$ whose pedal equation is $p=r\sin\alpha$ where α is constant *i.e.* p varies as r].

Again in order to find the Law, $P=\frac{h^2}{p^3}\cdot\frac{dp}{dr}$.

$$\therefore \quad P=\frac{h^2}{A^3r^3}\cdot A=\frac{h^2}{A^2 \cdot r^3}=\frac{k}{r^3}.$$

Hence the law is that of inverse cube.

Ex. 24. *Show that the only law of attraction for which the velocity in a circle at any distance is equal to the velocity from infinity to that distance is that of inverse cube.* **(Agra 65)**

Let us assume that the law of force is $\frac{\mu}{x^n}$.

Proceeding as in Q. 10, we have

$$v\frac{dv}{dx}=-P=-\frac{\mu}{x^n}. \quad ...(1)$$

If V be the velcoity acquired in falling from infinity to a distance a, then integrating (1) between the above limits, we get

$$\int_0^V v\,dv=\int_\infty^a -\frac{\mu}{x^n}\,dx$$

or

$$\frac{V^2}{2}=\left[\frac{\mu}{(n-1)(x^{n-1})}\right]_\infty^a=\frac{\mu}{n-1}\cdot\frac{1}{a^{n-1}}.$$

$$\therefore \quad V^2=\frac{2\mu}{n-1}\cdot\frac{1}{a^{n-1}}. \quad ..(1)$$

In the case of circle if velocity at a distance a be v, then

$$\frac{v^2}{a}=P=\frac{\mu}{a^n}; \quad \therefore \quad v^2=\frac{\mu}{a^{n-1}}. \quad ...(2)$$

We are given that $V=v$ and hence from (1) and (2), we get $\frac{2\mu}{n-1}\frac{1}{a^{n-1}}=\frac{\mu}{a^{n-1}}$ or $\frac{2}{n-1}=1$ or $2=n-1$ or $n=3$.

Hence the law of force is

$$\frac{\mu}{x^3} \text{ i.e. } \frac{\mu}{r^3} \text{ or } \mu u^3 \text{ i.e. inverse cube.}$$

Ex. 25. *A particle is acted on by a central repulsive force* $\frac{\mu r}{(r^2-9c^2)^2}$. *It is projected from an apse at a distance c*

with velocity $\sqrt{\frac{\mu}{8c^2}}$. *Show that it will describe a three-cusped hypo-cycloid and that the time to the cusp is*

$$\frac{4}{3}\pi c^2\sqrt{\frac{2}{\mu}}.$$

Since there is repulsive force, we have $P=-\frac{\mu r}{(r^2-9c^2)^2}$.

Now we know that in a central orbit $P=\frac{h^2}{p^3}\frac{dp}{dr}$.

$$\therefore\ \frac{h^2}{p^3}\frac{dp}{dr}=-\frac{\mu r}{(r^2-9c^2)^2}$$

or
$$\frac{2h^2}{p^3}\,dp=-\mu\,.\,\frac{2r}{(r^2-9c^2)^2}\,dr.$$

Integrating,
$$-\frac{h^2}{p^2}=-\frac{\mu}{-(r^2-9c^2)}+A$$

or
$$\frac{h^2}{p^2}=-\frac{\mu}{(r^2-9c^2)}-A.$$

But $vp=h$

$$\therefore\ \frac{h^2}{p^2}=v^2=-\frac{\mu}{(r^2-9c^2)}-A. \quad ...(1)$$

In order to find the value of A, we use the initial condition.

At an apse, $p=r=c$ and $v^2=\frac{\mu}{8c^2}$ given.

$$\therefore\ \frac{h^2}{c^2}=\frac{\mu}{8c^2}=-\frac{\mu}{-8c^2}-A;$$

$$\therefore\ A=0 \text{ and } h^2=\frac{\mu}{8}. \quad ..(2)$$

Hence from (1), we get

$$\frac{\mu}{8p^2}=\frac{-\mu}{(r^2-9c^2)} \quad \text{or} \quad 8p^2=9c^2-r^2. \quad ...(3)$$

Above is pedal equation of the curve known as three-cusped hypocycloid.

Now we have to find the time to reach the cusp. At the cusp $p=0$.

$\therefore$ from (3), we get $r^2-9c^2=0$ or $r=3c$.

Hence we have to find the time from $r=c$ to $r=3c$.

We know $vp=h$. But $v=\dfrac{ds}{dt}$.

$$\therefore \quad p.\frac{ds}{dt}=h \quad \text{or} \quad h\,dt=p.\frac{ds}{dr}\,dr$$

or
$$h\,dt=p\sec\phi\,dr.$$

$\because$ $\tan\phi=r\dfrac{d\theta}{dr}$, $\sin\phi=r\dfrac{d\theta}{ds}$, $\cos\phi=\dfrac{dr}{ds}$ and $p=r\sin\phi$.

$$\therefore \quad h\,dt=p\frac{1}{\sqrt{(1-\sin^2\phi)}}\,dr=p.\frac{1}{\sqrt{\left(1-\frac{p^2}{r^2}\right)}}\,dr$$

$$=\frac{p.r}{\sqrt{(r^2-p^2)}}\,dr.$$

Now put $p^2=\dfrac{9c^2-r^2}{8}$ from (3) in the above.

$$\therefore \quad \int_0^t h\,dt=\int_c^{3c} r.\frac{\sqrt{(9c^2-r^2)}}{\sqrt{(9r^2-9c^2)}}\,dr$$

$$=\int_c^{3c}\frac{r}{3}\sqrt{\left\{\frac{8c^2-(r^2-c^2)}{r^2-c^2}\right\}}\,dr.$$

Put $r^2-c^2=8c^2\sin^2\theta$; $\therefore$ $2r\,dr=8c^2.2\sin\theta\cos\theta\,d\theta$.

Also when $r=c$, $\sin\theta=0$; $\therefore$ $\theta=0$,

When $r=3c$, $\sin^2\theta=1$; $\therefore$ $\theta=\dfrac{\pi}{2}$.

$$ht=\int_0^{\pi/2}\frac{8c^2}{3}\sin\theta\cos\theta.\frac{\cos\theta}{\sin\theta}\,d\theta$$

$$=\frac{8c^2}{3}\int_c^{\pi/2}\cos^2\theta\,d\theta$$

or
$$ht=\frac{8c^2}{3}.\frac{1}{2}.\frac{\pi}{2}=\frac{2}{3}\pi c^2.$$

But from (2), $h=\sqrt{\frac{\mu}{8}}=\frac{\sqrt{\mu}}{2\sqrt{2}}$.

$$\therefore \quad t=\frac{2}{3}\pi c^2 \cdot \frac{2\sqrt{2}}{\sqrt{\mu}}=\frac{4}{3}\pi c^2\sqrt{\frac{2}{\mu}}.$$

Ex. 26. (a) *A particle of mass m is attached to a fixed point by an elastic string of natural length a, the coefficient of elasticity being nmg. It is projected from an apse at a distance a with velocity* $\sqrt{(2pgh)}$. *Show that the other apsidal distance is given by the equation*

$$nr^2(r-a)-2pha(r+a)=0. \qquad \text{(Vikram 65)}$$

If r be length of the string (stretched) in any position, then by Hooke's law,

$$\text{tension in the string}=\lambda\ \frac{\text{(Extension)}}{\text{Natural length}}$$

or
$$T=\lambda\ \frac{(r-a)}{a}=nmg\ \frac{(1-au)}{au}$$

which will act towards the centre.

Hence the acceleration towards the centre is

$$\frac{T}{m}=ng\ \frac{(1-au)}{au} \text{ which is equal to } P.$$

$$\therefore \quad h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P=ng\,\frac{(1-au)}{au}$$

or
$$h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\frac{ng}{a}\left(\frac{1}{u^3}-\frac{a}{u^2}\right).$$

Multiplying both sides by $2\,\frac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{ng}{a}\left[-\frac{1}{u^2}+\frac{2a}{u}\right]+A. \qquad \ldots(1)$$

In order to find A, we use the initial conditions, *i.e.* at an apse, $u=\frac{1}{a}$, $\frac{du}{d\theta}=0$ and $v^2=2pgh$ (given).

$$\therefore \quad 2pgh = h^2 \cdot \frac{1}{a^2} = \frac{ng}{a} \cdot a^2 + A \, .$$

$$\therefore \quad h^2 = 2pgha^2 \text{ and } A = 2pgh - nga.$$

Putting for h^2 and A in (1), we get

$$2pgha^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{ng}{a}\left[-\frac{1}{u^2}+\frac{2a}{u}\right]+2pgh-nga \quad \ldots(2)$$

We are to find the other apsidal distance. At an apse we know that $\frac{du}{d\theta}=0$ and hence from (2), we get

$$2pgha^2 \cdot \left[\frac{1}{r^2}\right]=\frac{ng}{a}\left[-r^2+2ar\right]+2pgh-nga$$

or $$2pgh\,(r^2-a^2)-\frac{ngr^2}{a}\,(r^2-2ar+a^2)=0$$

or $$2pha\,(r-a)\,(r+a)-nr^2\,(r-a)^2=0.$$

Cancel the factor $r-a$ which corresponds to the apse $r=a$.

Hence the other apsidal distance is given by

$$2pha\,(r+a)-nr^2\,(r-a)=0$$

or $$nr^2\,(r-a)-2pha\,(r+a)=0.$$

(b) *A particle of mass m is attached to a fixed point on a smooth horizontal table by an elastic string of natural length a, whose coefficient of elasticity is mg, If it is projected with a velocity due to half the length of the string in a direction perpendicular to the string which is initially unstretched, prove that the apsidal distances are given by*

$$a^2\,(r^2-a^2)-r^2\,(r-a)^2=0.$$

Ex. 27. *A particle is attached to a fixed point on a smooth horizontal plane by an elastic string of natural length a. Initially the particle is at rest on the plane with the string just taut and it is projected horizontally in a direction perpendicular to the string with a kinetic energy equal to*

the potential energy of the string when its extension is $\frac{3a}{\sqrt{2}}$. *Prove that the second apsidal distance is equal to 3a.*

We know that initially when the string is just taut, there is no tension and when its extension is $\frac{3a}{\sqrt{2}}$, then the tension will be $\lambda\frac{3a/\sqrt{2}}{a}=\frac{3\lambda}{\sqrt{2}}$.

Also we know that potential energy in an elastic string
$=$mean of initial and final tensions $\times$ extension

$$=\frac{1}{2}\left[0+\frac{3\lambda}{\sqrt{2}}\right]\frac{3a}{\sqrt{2}}=\frac{9a\lambda}{4}.$$

Again if V be the velocity of projection, then K. E. is $\frac{1}{2}mV^2$.

By the given condition, $\frac{a\lambda}{4}=\frac{1}{2}mV^2$; $\therefore$ $V^2=\frac{9a\lambda}{2m}$ ...(1)

Now proceeding exactly as in Q. 26, we have

$$h^2u^2\left[u+\frac{d^2u}{d\theta^2}\right]=P=\frac{T}{m}=\frac{\lambda}{m}\cdot\frac{r-a}{a}=\frac{\lambda}{m}\cdot\frac{1-au}{au}.$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{\lambda}{am}\left[-\frac{1}{u^2}+\frac{2a}{u}\right]+A. \quad ...(2)$$

In order to find A, we use the initial condition, *i.e.* when $u=\frac{1}{a}$, $\frac{du}{d\theta}=0$ and $v^2=\frac{9a\lambda}{2m}$ by (1).

$$\therefore \quad \frac{9a\lambda}{2m}=h^2\cdot\frac{1}{a^2}=\frac{\lambda}{am}[a^2]+A.$$

$$\therefore \quad h^2=\frac{9a^3\lambda}{2m} \text{ and } A=\frac{7a\lambda}{2m}.$$

Putting for h^2 and A in (2), we get

$$\frac{9a^3\lambda}{2m}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{\lambda}{am}\left[-\frac{1}{u^2}+\frac{2a}{u}\right]+\frac{7a\lambda}{2m}.$$

At an apse $\frac{du}{d\theta}=0$ and hence the apsidal distances are given by

$$\frac{9a^3}{2}.u^2=-\frac{1}{au^2}+\frac{2}{u}+\frac{7a}{2}$$

or $$9a^4u^4-7a^2u^3-4au+2=0$$

or $$2r^4-4ar^3-7a^2r^2+9a^4=0$$

or $$(r-a)(2r^3-2ar^2-9a^2r-9a^3)=0$$

or $$(r-a)(r-3a)(2r^2+4ar+3a^2)=0.$$

The last factor gives imaginary values of r.

Hence the two apsidal distances are $r=a$ and $r=3a$.

Ex. 28. *A particle is attached to a fixed point by a slightly elastic string and is projected at right angles to the string. Prove that the path is approximately*

$$r=c+c'\sin^2\left[\theta\sqrt{\left(\frac{c}{2c'}\right)}\right],$$

where c is the natural length and $c+c'$ is the greatest length of the string, which it attains in the motion.

Ex. 29. *A particle moves under a central attractive force*

$$\mu u^2(au+\sin\theta)\{1+2(au+\sin\theta)^2\}$$

Show that if it be projected at right angles to the radius vector at a distance a with velocity $\sqrt{\frac{\mu}{a}}$, it will pass through the centre after describing an angle of one radian about the centre.

$$h^2u^2\left[u+\frac{d^2u}{d\theta^2}\right]=\mu u^2(au+\sin\theta)[1+2(au+\sin\theta)^2].$$

But we know that $vp=h$ i.e. constant.

$$\therefore\quad \frac{\mu}{a}.a^2=h^2 \quad\text{or}\quad h^2=a\mu,$$

$$\therefore\quad u+\frac{d^2u}{d\theta^2}=\frac{1}{a}[(au+\sin\theta)+2(au+\sin\theta)^3]. \quad ...(1)$$

Put $au+\sin\theta=z$, $\therefore$ $a\dfrac{du}{d\theta}+\cos\theta=\dfrac{dz}{d\theta}$

or $\dfrac{d^2z}{d\theta^2}=a\dfrac{d^2u}{d\theta^2}-\sin\theta$, $\therefore$ $\dfrac{d^2z}{d\theta^2}+z=a\left(u+\dfrac{d^2u}{d\theta^2}\right)$.

Hence from (1), we get

$$\frac{1}{a}\left[\frac{d^2z}{d\theta^2}+z\right]=\frac{1}{a}\left[z+2z^3\right]$$

$$\frac{d^2z}{d\theta^2}=2z^3.$$

Multiplying both sides by $2\dfrac{dz}{d\theta}$ and integrating,

$$\left(\frac{dz}{d\theta}\right)^2=z^4+B. \qquad \text{...(2)}$$

When $u=\dfrac{1}{a}$, $\theta=0$, $\dfrac{du}{d\theta}=0$; $\therefore$ $z=a.\dfrac{1}{a}+0=1$

and $\dfrac{dz}{d\theta}=a.0+\cos 0=1$ and hence from (2), $B=0$.

$$\therefore \quad \frac{dz}{d\theta}=z^2 \quad \text{or} \quad \frac{dz}{z^2}=d\theta$$

or $\dfrac{1}{z}=-\theta+c$; when $\theta=0$, $z=1$, $\therefore$ $c=1$.

$$\therefore \quad \frac{1}{z}=1-\theta \quad \text{or} \quad z=\frac{1}{1-\theta}$$

or $au+\sin\theta=\dfrac{1}{1-\theta}$ or $\dfrac{a}{r}=\dfrac{1-\sin\theta\,(1-\theta)}{1-\theta}$.

$$\therefore \quad r=\frac{a\,(1-\theta)}{1-(1-\theta)\sin\theta}.$$

Now $r=0$ when $1-\theta=0$ *i.e.* $\theta=1$.

Ex. 30. *Show that if the central force is* $\dfrac{\mu r}{x^3}$, *the orbit is a conic and show that if the particle is projected at right angles to the radius vector at* $(a, 0)$ *with speed given by* $2v^2a=\mu$, *the orbit is a circle.*

$$h^2u^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{\mu r}{x^3}=\frac{\mu r}{r^3\cos^3\theta}=\mu u^2\sec^3\theta.$$

$$\therefore\quad u+\frac{d^2u}{d\theta^2}=\frac{\mu}{h^2}\sec^3\theta. \qquad ...(1)$$

Multiplying both sides by $\sin\theta$, we get

$$\sin\theta.\frac{d^2u}{d\theta^2}+u\sin\theta=\frac{\mu}{h^2}.\sec^2\theta.\frac{\sin\theta}{\cos\theta}.$$

Integrating w. r. t. θ, we get

$$\sin\theta.\frac{du}{d\theta}-\int\cos\theta.\frac{du}{d\theta}+\int u\sin\theta\,d\theta=\frac{\mu}{h^2}\frac{\tan^2\theta}{2}+A$$

or $$\sin\theta\frac{du}{d\theta}-u\cos\theta+\int u(-\sin\theta)\,d\theta+\int u\sin\theta\,d\theta$$

$$=\frac{\mu}{h^2}\frac{\tan^2\theta}{2}+A$$

or $$\sin\theta\frac{du}{d\theta}-u\cos\theta=\frac{\mu}{h^2}\frac{\tan^2\theta}{2}+A. \qquad ...(2)$$

Similarly multiplying both sides of (1) by $\cos\theta$ and integrating as above, we get

$$\cos\theta\frac{du}{d\theta}+u\sin\theta=\frac{\mu}{h^2}\tan\theta+B. \qquad ...(3)$$

In order to eliminate $\frac{du}{d\theta}$, multiply (3) by $\sin\theta$ and (2) by $\cos\theta$ and subtract.

$$\therefore\quad u=\frac{\mu}{h^2}\tan\theta\left(\sin\theta-\frac{1}{2}\tan\theta\cos\theta\right)+B\sin\theta-A\cos\theta. \qquad ...(4)$$

Converting (4) into cartesian by writing $\frac{x}{r}=\cos\theta$, $\frac{y}{r}=\sin\theta$,

we get $$\frac{1}{r}=\frac{\mu}{h^2}\frac{y}{x}\left(\frac{1}{2}.\frac{y}{r}\right)+B.\frac{y}{r}-A.\frac{x}{r}$$

or $$x=\frac{\mu}{2h^2}y^2+Bxy-Ax^2.$$

Above being a general equation of 2nd degree represents in general a conic section.

In the second case the particle is projected at right angles to the radius vector at $(a, 0)$ *i. e.* $\theta=0$ and $p=a$,

$\therefore \quad vp=h$ or $v^2p^2=h^2$ or $\frac{\mu}{2a}\cdot a^2=h^2$ or $\frac{\mu}{2h^2}=\frac{1}{a}$.

Putting for h^2 in (4), we get

$$u=\frac{1}{a}\,\frac{\sin^2\theta}{\cos\theta}+B\sin\theta-A\cos\theta. \qquad \ldots(5)$$

In order to find A and B, we use the initial conditions that when $\theta=0$, $r=a$ *i. e.* $u=\frac{1}{a}$ and $\frac{du}{d\theta}=0$.

Then from (2), $-\frac{1}{a}=0+A$; $\quad \therefore \quad A=-\frac{1}{a}$,

and from (3), $\qquad 0=0+B$; $\quad \therefore \quad B=0$.

Putting for A and B in (5), we get the equation of the path as $\frac{1}{r}=\frac{1}{a}\,\frac{\sin^2\theta}{\cos\theta}+\frac{1}{a}\cos\theta=\frac{1}{a\cos\theta}\cdot 1$ or $\quad r=a\cos\theta$, which is the standard polar equation of a circle whose diameter is of length a and passes through the pole.

§ 7. *Stability of an orbit—nearly circular orbit.*

Stable orbit. If a particle describing an orbit is slightly disturbed and it still describes a path which does not deviate far from the original orbit, then the orbit is said to be stable.

A particle describes a path which is nearly a circle about a centre of force ($=\mu u^n$) at its centre ; find the condition that this may be a stable motion.

$$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P=\mu u^n$$

or
$$h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu u^{n-2}. \qquad \ldots(1)$$

If the path is a circle of radius $\frac{1}{c}$ *i. e.* $u=c$, then $\frac{d^2u}{d\theta^2}=0$.

$\therefore$ from (1), we get $h^2c=\mu c^{n-2}$ or $h^2=\mu c^{n-3}$.

Putting in (1), we get $u+\frac{d^2u}{d\theta^2}=\frac{1}{c^{n-3}}\, u^{n-2}$. ...(2)

Suppose that the circular motion is slightly disturbed in such a way that h remains unchanged, *e. g.* it be given a small additional velocity in a direction away from the centre of force by means of a blow so that there is a change in the radial velocity and transverse veloctiy remains unaltered. It may be noted that since $r\frac{d\theta}{dt}$ is unchanged, $r^2\frac{d\theta}{dt}=h$ is also unchanged.

Putting $u=c+x$ where x is small in (2), we get

$$c+x+\frac{d^2x}{d\theta^2}=\frac{1}{c^{n-3}}(c+x)^{n-2}=\frac{c^{n-2}}{c^{n-3}}\left(1+\frac{x}{c}\right)^{n-2}$$

or $$c+x+\frac{d^2x}{d\theta^2}=c\left[1+(n-2).\frac{x}{c}+\ldots\right]=c+(n-2)\,x.$$

$$\therefore\quad \frac{d^2x}{d\theta^2}=(n-3)\,x=-(3-n)\,x\quad \text{(Note)}.\qquad ...(3)$$

If n be <3, then $3-n$ is +ive and the above equation is a S. H. M. and its solution is

$$x=A\cos[\sqrt{(3-n)}\,\theta+B].$$

$\left[\text{Solution of } \frac{d^2y}{dx^2}+a^2y=0 \text{ is } y=c_1\cos(ax+c_2).\right]$

$$\therefore\quad u=c+x=c+A\cos[\sqrt{(3-n)}\,\theta+B].\qquad ...(4)$$

As θ increases x decreases and it is small so that the orbit continues to be nearly circular.

However if $n>3$ so that $(n-3)$ is +ive, then the solution of $\frac{d^2x}{d\theta^2}=(n-3)\,x$ is

$$x=Ae^{\sqrt{(n-3)}\,\theta}+Be^{-\sqrt{(n-3)}\,\theta}\text{ and } u=c+x.$$

$\left[\text{Solution of } \frac{d^2y}{dx^2} - a^2y = 0 \text{ is } y = c_1e^{ax} + c_2e^{-ax}\right].$

Here as θ increases x also increases so that x is not always small and the orbit does not continue to be nearly circular.

Hence when $n < 3$, $u = c + A\cos[\sqrt{(3-n)}\,\theta + B]$ is the approximation to the path and also from (3) it is clear that the motion is S. H. M. (or stable).

Apsidal distances. The apses are given by $\frac{du}{d\theta} = 0$.

$$\therefore \quad \sin[\sqrt{(3-n)}\,\theta + B)] = 0.$$

Hence $\sqrt{(3-n)}\,\theta + B = r\pi$, where $r = 0, 1, 2, 3, \ldots$ and the corresponding values of θ be $\theta_0, \theta_1, \theta_2, \ldots$

$$\therefore \quad \sqrt{(3-n)}\,\theta_0 + B = 0, \ \sqrt{(3-n)}\,\theta_1 + B = \pi,$$
$$\sqrt{(3-n)}\,\theta_2 + B = 2\pi, \ldots$$

$$\therefore \quad \theta_1 - \theta_0 = \theta_2 - \theta_1 = \theta_3 - \theta_2 = \ldots = \frac{\pi}{\sqrt{(3-n)}}.$$

Above shows that the difference between the successive values of θ at the apses is $\frac{\pi}{\sqrt{(3-n)}}$. Hence the apsidal angle for nearly circular path is $\frac{\pi}{\sqrt{(3-n)}}$.

Since the Max. and Min. values of cos are $+1$ and -1 respectively, hence we observe that Max. and Min. values of u are $C+A$ and $C-A$ respectively. In other words apses are $C+A$ and $C-A$.

8. General case.

Above we have discussed the case when $P = \mu u^n$. Below we discuss the general case when the central acceleration is $\phi(u)$, *i.e.* $P = \phi(u)$.

$$h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = P = \phi(u)$$

or $$u+\frac{d^2u}{d\theta^2}=\frac{\phi(u)}{h^2u^2}.$$

For a circle of radius $\frac{1}{c}$, *i.e.* $u=c$ (constant), $\frac{d^2u}{d\theta^2}=0.$

$$\therefore \quad c=\frac{\phi(c)}{h^2c^2}; \quad \therefore \ h^2=\frac{\phi(c)}{c^3}.$$

$$\therefore \quad u+\frac{d^2u}{d\theta^2}=\frac{c^3}{\phi(c)}\ \frac{\phi(u)}{u^2}. \qquad \ldots(1)$$

Putting $u=c+x$ in (1), we get

$$c+x+\frac{d^2x}{d\theta^2}=\frac{c^3}{\phi(c)}\ \frac{\phi(c+x)}{(c+x)^2}$$

or $$c+x+\frac{d^2x}{d\theta^2}=\frac{c^3}{\phi(c)}\cdot\frac{[\phi(c)+x\phi'(c)+..]}{c^2\left[1+\frac{2x}{c}+\ldots\right]}$$

$$=\frac{c}{\phi(c)}[\phi(c)+x\phi'(c)+\ldots]\left[1-\frac{2x}{c}\ldots\right]$$

$$=\frac{c}{\phi(c)}\left[\phi(c)+x\phi'(c)-\frac{2x}{c}\phi(c)\ldots\right]$$

$$=c-2x+c\frac{\phi'(c)}{\phi(c)}x+\ldots$$

or $$c+x+\frac{d^2x}{d\theta^2}=c-2x+c\frac{\phi'(c)}{\phi(c)}x.$$

$$\therefore \quad \frac{d^2x}{d\theta^2}=-\left(3-c\ \frac{\phi'(c)}{\phi(c)}\right)x. \qquad \ldots(2)$$

The above is S. H. M., *i.e.* stable only if $3-c\ \frac{\phi'(c)}{\phi(c)}$ is +ive or $c\ \frac{\phi'(c)}{\phi(c)}$ is <3.

It can easily be shown as in the particular case that the apsidal angle is $\frac{\pi}{\sqrt{\left[3-c\ \frac{\phi'(c)}{\phi(c)}\right]}}$.

Differentiating both sides w. r. t. θ,

$$\frac{d}{d\theta}\left\{\frac{\frac{P}{u^2}-\frac{T}{u^3}\frac{du}{d\theta}}{u+\frac{d^2u}{d\theta^2}}\right\}=\frac{d}{d\theta}(h^2)=2h.\frac{dh}{d\theta}=\frac{2T}{u^3} \text{ from (}$$

Ex. 1. *One end of an elastic string of unstretched length a is tied to a point on the top of a smooth table, and a point attached to the other end can move freely on the table. If the path be nearly circular of radius b, show that its apsidal angle is approximately* $\pi\sqrt{\left(\frac{b-a}{4b-3a}\right)}$.

If r be the stretched length of the string, then by Hooke's Law tension in the string is

$$\lambda\,\frac{(r-a)}{a}=\lambda\,\frac{(1-au)}{au}=P.$$

Hence the equation of motion is

$$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P=\lambda\,\frac{(1-au)}{au}$$

or
$$u+\frac{d^2u}{d\theta^2}=\frac{\lambda}{ah^2}\left[\frac{1}{u^3}-\frac{a}{u^2}\right]. \qquad \ldots(1)$$

If the path is a circle of radius b, then $r=b$

or
$$u=\frac{1}{b}\,; \quad \therefore \quad \frac{d^2u}{d\theta^2}=0.$$

Hence from (1), we get

$$\frac{1}{b}=\frac{\lambda}{ah^2}\,[b^3-ab^2]\,; \quad \therefore \quad h^2=\frac{\lambda}{a}.b^3\,(b-a). \qquad \ldots(2)$$

Now let the particle be displaced slightly is such a way that h^2 remains constant and the path be nearly circular.

Putting $u=\frac{1}{b}+x$ where x is small in (1), we get

$$\frac{1}{b}+x+\frac{d^2x}{d\theta^2}=\frac{\lambda}{ah^2}\left[\frac{b^3}{(1+bx)^3}-a\,.\frac{b^2}{(1+bx)^2}\right]$$

§ 9. Differential Equation of the path under radial and transverse accelerations. (Cal. Hons. 59)

Now let us suppose that in addition to the central acceleration ; P (*i.e.* radial acceleration is $-P$), we have an acceleration T perpendicular to P (*i.e.* transverse acceleration is T); then the equations of motion are as under :

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=-P, \qquad ...(1)$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=T. \qquad ...(2)$$

Let us put $r^2\frac{d\theta}{dt}=h$ which is not constant here.

Hence from (2), we get $u\frac{dh}{dt}=T$ ...(3)

or
$$T=u.\frac{dh}{d\theta}.\frac{d\theta}{dt}=u.\frac{dh}{d\theta}.\frac{h}{r^2}=u^3h\frac{dh}{d\theta}. \qquad ...(4)$$

$$\frac{dr}{dt}=\frac{dr}{d\theta}.\frac{d\theta}{dt}=\left(-\frac{1}{u^2}\frac{du}{d\theta}\right).\frac{h}{r^2}=-h\frac{du}{d\theta}, \quad \because \quad u=\frac{1}{r}.$$

$$\frac{d^2r}{dt^2}=-\left[h\frac{d^2u}{d\theta^2}.\frac{d\theta}{dt}+\frac{du}{d\theta}.\frac{dh}{dt}\right]$$

$$=-\left[h\frac{d^2u}{d\theta^2}.\frac{h}{r^2}+\frac{T}{u}.\frac{du}{d\theta}\right] \text{ from (3).}$$

Putting the above results in (1), we get

$$-h^2u^2\frac{d^2u}{d\theta^2}-\frac{T}{u}\frac{du}{d\theta}-\frac{1}{u}.h^2u^4=-P$$

or
$$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P-\frac{T}{u}\frac{du}{d\theta}$$

or
$$u+\frac{d^2u}{d\theta^2}=\frac{P-\frac{T}{u}\frac{du}{d\theta}}{h^2u^2}$$

or
$$\frac{\frac{P}{u^2}-\frac{T}{u^3}\frac{du}{d\theta}}{u+\frac{d^2u}{d\theta^2}}=h^2.$$

Differentiating both sides w. r. t. θ,

$$\frac{d}{d\theta}\left\{\frac{\frac{P}{u^2}-\frac{T}{u^3}\frac{du}{d\theta}}{u+\frac{d^2u}{d\theta^2}}\right\}=\frac{d}{d\theta}(h^2)=2h\cdot\frac{dh}{d\theta}=\frac{2T}{u^3} \text{ from }$$

Ex. 1. *One end of an elastic string of unstretched length a is tied to a point on the top of a smooth table, and a point attached to the other end can move freely on the table. If the path be nearly circular of radius b, show that its apsidal angle is approximately* $\pi\sqrt{\left(\frac{b-a}{4b-3a}\right)}$.

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Hence the equation of motion is

$$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P=\lambda\,\frac{(1-au)}{au}$$

or
$$u+\frac{d^2u}{d\theta^2}=\frac{\lambda}{ah^2}\left[\frac{1}{u^3}-\frac{a}{u^2}\right]. \qquad \ldots(1)$$

If the path is a circle of radius b, then $r=b$

or
$$u=\frac{1}{b}\,; \quad \therefore \quad \frac{d^2u}{d\theta^2}=0.$$

Hence from (1), we get

$$\frac{1}{b}=\frac{\lambda}{ah^2}[b^3-ab^2]\,; \quad \therefore \quad h^2=\frac{\lambda}{a}\cdot b^3\,(b-a). \qquad \ldots(2)$$

Now let the particle be displaced slightly is such a way that h^2 remains constant and the path be nearly circular.

Putting $u=\frac{1}{b}+x$ where x is small in (1), we get

$$\frac{1}{b}+x+\frac{d^2x}{d\theta^2}=\frac{\lambda}{ah^2}\left[\frac{b^3}{(1+bx)^3}-a\cdot\frac{b^2}{(1+bx)^2}\right]$$

Hence the apsidal angle is $\pi\div\sqrt{\left(1-\frac{3m^2}{n^2}\right)}$

$$=\pi\left(1-\frac{3m^2}{n^2}\right)^{-1/2}=\pi\left(1+\frac{3}{2}\frac{m^2}{n^2}\right) \text{ approximately}$$

after neglecting cubes and higher powers.

Ex. 3. *A particle is moving in a circular orbit of radius a under a force $\mu u^3(2a^2u^2-1)$ towards the centre. Show that the orbit is unstable and if slightly disturbed, it will take the form* $r=a\tanh\theta$ *or* $r=a\coth\theta$.

$$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=\mu u^3(2a^2u^2-1).$$

$$\therefore\quad\left(u+\frac{d^2u}{d\theta^2}\right)=\frac{\mu}{h^2}(2a^2u^3-u).\qquad\ldots(1)$$

For a circle of radius a, $u=\frac{1}{a}$; $\therefore\ \frac{d^2u}{d\theta^2}=0$.

$$\therefore\quad\frac{1}{a}=\frac{\mu}{h^2}\cdot\left(\frac{2a^2}{a^3}-\frac{1}{a}\right)=\frac{\mu}{h^2}\cdot\frac{1}{a};\quad\therefore\quad\frac{\mu}{h^2}=1.$$

$$\therefore\quad u+\frac{d^2u}{d\theta^2}=(2a^2u^3-u)\text{ by (1)}.\qquad\ldots(2)$$

Putting $u=\frac{1}{a}+x$ where $x<1$ in (2), we get

$$\frac{1}{a}+x+\frac{d^2x}{d\theta^2}=2a^2\cdot\frac{(1+ax)^3}{a^3}-\left(\frac{1}{a}+x\right)$$

$$=\frac{2}{a}(1+3ax+\ldots)-\frac{1}{a}-x$$

$$=\frac{1}{a}+6x-x.$$

$\therefore\ \frac{d^2x}{d\theta^2}=4x$ which is not S. H. M.; hence unstable.

In order to find the path we multiply both sides of (2) by $2\frac{du}{d\theta}$ and integrate.

$$u^2+\left(\frac{du}{d\theta}\right)^2=a^2u^4-u^2+A. \qquad \ldots(3)$$

For a circle $u=\frac{1}{a},\ \frac{du}{d\theta}=0$; $\therefore\ A=\frac{1}{a^2}$.

$$\therefore\ \frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{a^2}{r^4}-\frac{1}{r^2}+\frac{1}{a^2}, \quad \because\ u=\frac{1}{r}.$$

$$\therefore\ \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{a^4-2a^2r^2+r^4}{a^2r^4}=\frac{(a^2-r^2)^2}{a^2r^4}.$$

$$\therefore\ \frac{a\,dr}{(a^2-r^2)}=\pm d\theta.$$

If $r>a$ we write $\frac{a\,dr}{r^2-a^2}=d\theta$ or $\coth^{-1}\frac{r}{a}=\theta+B$

or $$r=a\coth(\theta+B).$$

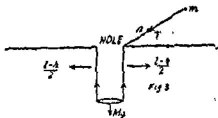
If $r<a$ we write $\frac{a\,dr}{a^2-r^2}=d\theta$ or $\tanh^{-1}\frac{r}{a}=\theta+C$

or $$r=a\tanh(\theta+C).$$

In order to put the results in the given form, we have to take the initial conditions properly.

Ex. 4. *A particle of mass m can move on a smooth horizontal table. It is attached to a string which passes through a smooth hole in the table, goes under a small smooth pulley of mass M and is attached to a point in the under side of the table so that the parts of the string hang vertically. If the motion be slightly disturbed when m is describing a circle uniformly, so that the angular momentum is unchanged, prove that the apsidal angle is*

$$\pi\sqrt{\left(\frac{M+4m}{12m}\right)}.$$



The particle of mass m is describing a central orbit under the

tension T towards the hole. If P be the attraction towards the centre, then $m.P=T$ or $P=\frac{T}{m}$. Hence the equation of motion is $h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=P=\frac{T}{m}$

or $$m\left(u+\frac{d^2u}{d\theta^2}\right)=\frac{T}{h^2u^2}, \quad \ldots(1)$$

where $$r^2\frac{d\theta}{dt}=h \quad \text{or} \quad \frac{d\theta}{dt}=\frac{h}{r^2}=hu^2. \quad \ldots(2)$$

If l be the length of the string, then the portion on either side of pulley is $\frac{l-r}{2}$. The equation of motion for M moving vertically downwards is

$$M\frac{d^2}{dt^2}\left(\frac{l-r}{2}\right)=Mg-2T \quad \text{or} \quad -\frac{1}{2}M\frac{d^2r}{dt^2}=Mg-2T \quad \ldots(3)$$

Now $\frac{dr}{dt}=\frac{d}{dt}\left(\frac{1}{u}\right)=-\frac{1}{u^2}\frac{du}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}\cdot hu^2$

or $\frac{dr}{dt}=-h\frac{du}{d\theta}$ by (2); $\therefore$ $\frac{d^2r}{dt^2}=-h\frac{d^2u}{d\theta^2}\cdot\frac{d\theta}{dt}=-h^2u^2\frac{d^2u}{d\theta^2}$ by (2).

Hence from (3) on putting for $\frac{d^2r}{dt^2}$, we get

$$-\frac{1}{2}M\left(-h^2u^2\frac{d^2u}{d\theta^2}\right)=Mg-2T$$

or $$\frac{M}{4}\frac{d^2u}{d\theta^2}=\frac{Mg}{2h^2u^2}-\frac{T}{h^2u^2}. \quad \ldots(4)$$

Adding (1) and (4) thereby eliminating T, we get

$$mu+\left(m+\frac{M}{4}\right)\frac{d^2u}{d\theta^2}=\frac{Mg}{2h^2u^2}$$

or $$u+\frac{4m+M}{4m}\frac{d^2u}{d\theta^2}=\frac{Mg}{2mh^2u^2}. \quad \ldots(5)$$

If the path is a circle of radius $\frac{1}{c}$, then $u=c$, $\frac{d^2u}{d\theta^2}=0.$

$$\therefore \quad c=\frac{Mg}{2mh^2c^2} \quad \text{or} \quad c^3=\frac{Mg}{2mh^2}.$$

Putting for h^2 in (5), we get

$$u+\frac{4m+M}{4m}\frac{d^2u}{d\theta^2}=\frac{c^3}{u^2}. \qquad \ldots(6)$$

Above gives us the equation of the circular path of m.

Let the particle be displaced slightly, so that h remains constant.

Putting $u=c+x$ in (6), we get

$$c+x+\frac{4m+M}{4m}\frac{d^2x}{d\theta^2}=\frac{c^3}{(c+x)^2}=\frac{c^3}{c^2(1+x/c)^2}$$

$$=c\left(1+\frac{x}{c}\right)^{-2}=c\left(1-\frac{2x}{c}+\ldots\right)$$

$$=c-2x.$$

$$\therefore \quad \frac{4m+M}{4m}\frac{d^2x}{d\theta^2}=-3x \quad \text{or} \quad \frac{d^2x}{d\theta^2}=-\frac{12m}{4m+M}x,$$

which is S. H. M. Hence the apsidal angle is

$$\pi\div\sqrt{\left(\frac{12m}{4m+M}\right)}=\pi\sqrt{\left(\frac{4m+M}{12m}\right)}.$$

Ex. 5. (a) *Two masses M, m are connected by a string which passes through a hole in a smooth horizontal plane, the mass m hanging vertically. Show that M describes on the plane a curve whose differential equation is*

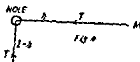
$$\left(1+\frac{m}{M}\right)\frac{d^2u}{d\theta^2}+u=\frac{mg}{M}\cdot\frac{1}{h^2u^2}.$$

Prove also that the tension of the string is

$$\frac{Mm}{M+m}(g+h^2u^3).$$

The equation of motion for M is

$$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=\frac{T}{M}. \qquad \ldots(1)$$



If l be the length of the string,

...ien the equation of motion for m hanging vertically is

$$m \frac{d^2}{dt^2}(l-r)=mg-T$$

or
$$\frac{d^2r}{dt^2}=\frac{T}{m}-g \quad \text{or} \quad -h^2u^2\frac{d^2u}{d\theta^2}=\frac{T}{m}-g, \qquad ...(2)$$

as on P. 96 in last Q.

Eliminating T between (1) and (2), we get

$$Mh^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=mg-mh^2u^2\frac{d^2u}{d\theta^2} \text{ (each equal to } T)$$

or
$$(M+m)\,h^2u^2\frac{d^2u}{d\theta^2}+Mh^2u^3=mg.$$

Dividing throughout by Mh^2u^2, we get

$$\left(1+\frac{m}{M}\right)\frac{d^2u}{d\theta^2}+u=\frac{mg}{M}\cdot\frac{1}{h^2u^2}. \qquad ...(3)$$

Above gives us the required differential equation of the path of M.

Now substituting the value of $\frac{d^2u}{d\theta^2}$ from (3) in (2), we get

$$T=mg-mh^2u^2\frac{d^2u}{d\theta^2}=mg-mh^2u^2\left[\frac{mg}{M+m}\cdot\frac{1}{h^2u^2}-\frac{Mu}{M+m}\right]$$

or
$$T=mg-\frac{m^2g}{M+m}+\frac{mMh^2u^3}{M+m}$$

$$=\frac{Mmg}{M+m}+\frac{Mmh^2u^3}{M+m}$$

or
$$T=\frac{Mm}{M+m}(g+h^2u^3).$$

(b) *In the part (a) if $m=M$ and the later be projected on the plane with velocity $\sqrt{\left(\frac{8ag}{3}\right)}$ from an apse at a distance a, show that the former will rise a distance a.*

Putting $m=M$ in (3) of part (a), we get

$$\left(1+\frac{M}{M}\right)\frac{d^2u}{d\theta^2}+u=\frac{Mg}{M}\,\frac{1}{h^2u^2}$$

or $$\frac{d^2u}{d\theta^2}+\frac{1}{2}u=\frac{g}{2h^2u^2}.$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$\left(\frac{du}{d\theta}\right)^2+\frac{1}{2}u^2=-\frac{g}{h^2u}+A. \quad ...(1)$$

Now $v=\sqrt{\left(\frac{8ag}{3}\right)}$ and $vp=h$ (constant).

$$\therefore \quad \sqrt{\left(\frac{8ag}{3}\right)}.a=h\,; \qquad \therefore \quad h^2=\frac{8a^3g}{3}.$$

Also when $u=\frac{1}{a}$, $\frac{du}{d\theta}=0$ at an apse.

$$\therefore \quad 0+\frac{1}{2}.\frac{1}{a^2}=-\frac{ag.3}{8a^3g}+A.$$

$$\therefore \quad \frac{1}{a^2}\left(\frac{1}{2}+\frac{3}{8}\right)=A \quad \text{or} \quad A=\frac{7}{8a^2}.$$

Hence the equation (1) on putting for h^2 and A becomes

$$\left(\frac{du}{d\theta}\right)^2+\frac{1}{2}u^2=-\frac{3}{8a^2u}+\frac{7}{8a^2}. \quad ...(3)$$

In order to find the apse we put $\frac{du}{d\theta}=0$ in (3).

$$\therefore \quad \frac{1}{2}.\frac{1}{r^2}=-\frac{3}{8a^3}r+\frac{7}{8a^2}$$

or $$4a^3=-3r^3+7ar^2 \quad \text{or} \quad 3r^3-7ar^2+4a^3=0$$

or $$(r-a)(r-2a)(3r+2a)=0.$$

Above gives $r=a$ and $2a$, showing that the other apsidal distance is $2a$. Hence we conclude that the mass on the table will move forward a distance $2a-a=a$ and consequently will raise the hanging mass through a distance a.

(c) *Two particles of masses M and m are connected by a light string, the string passes through a small hole in the*

table, m hangs vertically and M describes a curve on the table which is very nearly a circle whose centre is the hole. Show that the apsidal angle is $\pi\sqrt{\left(\frac{M+m}{3M}\right)}$.

From equation (3) of part (a), the differential equation of the path of M is

$$\left(1+\frac{m}{M}\right)\frac{d^2u}{d\theta^2}+u=\frac{mg}{M}\cdot\frac{1}{h^2u^2}. \quad ...(1)$$

If the path be a circle of radius $\frac{1}{c}$, then $u=c$; $\therefore \frac{d^2u}{d\theta^2}=0$.

$$\therefore \quad c=\frac{mg}{M}\cdot\frac{1}{c^2h^2}; \qquad \therefore \quad \frac{mg}{Mh^2}=c^3.$$

The equation (1) reduces to

$$\left(1+\frac{m}{M}\right)\frac{d^2u}{d\theta^2}+u=\frac{c^3}{u^2}. \quad ...(2)$$

Above gives us the equation of the circular path of M. Let the particle be disturbed slightly from the circular path so that h remains constant.

Putting $u=c+x$ in (2) where x is small, we get

$$\left(\frac{M+m}{M}\right)\frac{d^2x}{d\theta^2}+c+x=\frac{c^3}{(c+x)^2}=\frac{c^3}{c^2}\left(1+\frac{x}{c}\right)^{-2}$$

$$=c\left(1-\frac{2x}{c}+\ldots\right)=c-2x.$$

$$\therefore \quad \frac{d^2x}{d\theta^2}=-\frac{3M}{M+m}x, \quad \text{i.e. S.H.M.}$$

Apsidal angle is $\pi\div\sqrt{\left(\frac{3M}{M+m}\right)}=\pi\sqrt{\left(\frac{M+m}{3M}\right)}$.

Ex. 6. *Two particles of masses M and m are attached to the ends of an inextensible string which passes through a smooth fixed ring, the whole resting on a horizontal table. The particle is being projected at right angles to the string. Show that its path is*

$$a=r\cos\left[\sqrt{\left(\frac{m}{m+M}\right)}\,\theta\right].$$

If T be the tension in the string, then for the particle m, we have the following equations :—

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=-\frac{T}{m},\ \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=0.$$

$$\therefore\quad r^2\frac{d\theta}{dt}=\text{constant}=h\quad\text{or}\quad\frac{d\theta}{dt}=\frac{h}{r^2}.$$

$$\therefore\quad \frac{d^2r}{dt^2}-r.\frac{h^2}{r^4}=-\frac{T}{m}. \qquad ...(1)$$

Also for M, we have

$$\frac{d^2}{dt^2}(l-r)=-\frac{T}{M}\quad\text{or}\quad\frac{d^2r}{dt^2}=\frac{T}{M}. \qquad ...(2)$$

Eliminating T between (1) and (2), we get

$$\frac{d^2r}{dt^2}\left(1+\frac{M}{m}\right)=\frac{h^2}{r^3}.$$

Multiplying by $2\frac{dr}{dt}$ and integrating,

$$\left(1+\frac{M}{m}\right)\left(\frac{dr}{dt}\right)^2=-\frac{h^2}{r^2}+A. \qquad ...(2)$$

When $r=a$ initially $\frac{dr}{dt}=0$ as the particle is projected at right angles to the string.

$\therefore\quad A=\frac{h^2}{a^2}$. Putting for A in (3), we get

$$\left(1+\frac{M}{m}\right)\left(\frac{dr}{dt}\right)^2=h^2\left(\frac{1}{a^2}-\frac{1}{r^2}\right)=h^2\,\frac{r^2-a^2}{a^2r^2}.$$

or

$$\sqrt{\left(\frac{M+m}{m}\right)}.\frac{dr}{d\theta}.\frac{d\theta}{dt}=h\,\frac{\sqrt{(r^2-a^2)}}{ar}$$

or $$\sqrt{\left(\frac{M+m}{m}\right)}\cdot\frac{dr}{d\theta}\cdot\frac{h}{r^2}=h\frac{\sqrt{(r^2-a^2)}}{ar}, \quad \because \ r^2\frac{d\theta}{dt}=h$$

or $$\frac{a}{r\sqrt{(r^2-a^2)}}\,dr=\sqrt{\left(\frac{m}{M+m}\right)}\,d\theta$$

or $$\sec^{-1}\frac{r}{a}=\sqrt{\left(\frac{m}{M+m}\right)}\,\theta+B.$$

When $r=a$, $\theta=0$; $\therefore$ $B=0$.

$$\therefore \quad \cos^{-1}\frac{a}{r}=\sqrt{\left(\frac{m}{M+m}\right)}\,\theta, \quad \because \ \sec^{-1}x=\cos^{-1}\frac{1}{x}$$

or $$a=r\cos\sqrt{\left(\frac{m}{M+m}\right)}\,\theta$$

Ex. 7. *A particle moves in an orbit under a central acceleration $\frac{\mu}{r^2}$ along the radius vector. Obtain the equations of energy and angular momentum and show that if the particle is projected with velocity u at right angles to the radius at a distance c from the origin,*

$$\left(\frac{dr}{dt}\right)^2=\left\{\frac{2\mu}{c}-u^2\left(1+\frac{c}{r}\right)\right\}\left(\frac{c}{r}-1\right). \qquad \textbf{(Agra 1962)}$$

Let f be the force per unit of mass towards the centre of force. Then work done in small displacement δs is

$$-f\frac{dr}{ds}\,ds=-f\,dr=-\frac{\mu}{r^2}\,dr, \quad \because \ f=\frac{\mu}{r^2}.$$

Also radial velocity $=\dot{r}$ and transverse velocity is $r\dot{\theta}$.

Hence resultant velocity is $\sqrt{\{\dot{r}^2+(r\dot{\theta})^2\}}=V$.

$$\frac{1}{2}\,[\dot{r}^2+(r\dot{\theta})^2]=-\int f\,dr+A$$

or $$\frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2\right]=-\int\frac{\mu}{r^2}\,dr+A=\frac{\mu}{r}+A. \qquad ...(1)$$

Again we know in the case of central orbits,

$$r^2\frac{d\theta}{dt}=h=\text{constant}.$$

Hence the equation of angular momentum is

$$r^2 \frac{d\theta}{dt} = h. \qquad \ldots(2)$$

Equations (1) and (2) are the required equations.

In order to find the constants A and h, we use initial conditions.

The particle is projected at right angles to radius vector at a distance c from origin with velocity u, $\therefore$ $r=c$, radial velocity $\frac{dr}{dt}=0$ and transverse velocity $r\frac{d\theta}{dt}=u$.

Putting the above data in (1), we get

$$\frac{1}{2}[0+u^2]=\frac{\mu}{c}+A \quad \therefore \quad A=\frac{1}{2}u^2-\frac{\mu}{c}.$$

$$\therefore \quad \frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2\right]=\frac{\mu}{r}+\frac{u^2}{2}-\frac{\mu}{c}. \qquad \ldots(2)$$

Also from (2), we get $r.r\frac{d\theta}{dt}=h$ or $c.u=h$.

$$\therefore \quad r^2\frac{d\theta}{dt}=h=cu \quad \text{or} \quad \frac{d\theta}{dt}=\frac{cu}{r^2}. \qquad \ldots(4)$$

Eliminating $\frac{d\theta}{dt}$ between (3) and (4), we get

$$\frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2+r^2.\frac{c^2u^2}{r^4}\right]=\frac{\mu}{r}+\frac{u^2}{2}-\frac{\mu}{c}$$

or $$\left(\frac{dr}{dt}\right)^2=2\mu\left(\frac{1}{r}-\frac{1}{c}\right)+u^2-\frac{c^2u^2}{r^2}$$

or $$\left(\frac{dr}{dt}\right)^2=\frac{2\mu}{c}\left(\frac{c}{r}-1\right)-u^2\left(\frac{c^2}{r^2}-1\right)$$

or $$\left(\frac{dr}{dt}\right)^2=\left[\frac{2\mu}{c}-u^2\left(\frac{c}{r}+1\right)\right]\left(\frac{c}{r}-1\right). \qquad \text{Proved.}$$

Ex. 8. *A particle of mass m is projected under a central force $m\mu u^2$. If it be projected from the point $r=c$, $\theta=0$ at right angles to the radius vector with velocity $\left(\frac{n\mu}{c}\right)^{1/2}$. Prove*

that $$\left(r\frac{dr}{dt}\right)^2=\frac{\mu}{c}(r-c)\,[nc-(2-n)\,r].$$

(Delhi Hons. 65)

Proceeding exactly as above, we have

$$\frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2\right]=-\int\frac{\mu}{r^2}\,dr+A=\frac{\mu}{r}+A. \qquad ...(1)$$

Also $$r^2\frac{d\theta}{dt}=h. \qquad ...(2)$$

Initially, $r=c,\ \frac{dr}{dt}=0,\ r\frac{d\theta}{dt}=\sqrt{\left(\frac{n\mu}{c}\right)}$.

Putting the above data in (1) and (2), we get

$$\frac{1}{2}\left[0+\frac{n\mu}{c}\right]=\frac{\mu}{c}+A. \quad \therefore \quad A=\frac{\mu}{2c}(n-2).$$

Also $r^2\frac{d\theta}{dt}=h$ or $r.r\frac{d\theta}{dt}=h$ or $c.\sqrt{\left(\frac{n\mu}{c}\right)}=h$

$$\therefore \quad h=\sqrt{(n\mu c)}.$$

Putting for A and h in (1) and (2), we get

$$\frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2\right]=\frac{\mu}{r}+\frac{\mu}{2c}(n-2) \qquad ...(3)$$

and $$r^2\frac{d\theta}{dt}=\sqrt{(n\mu c)}. \qquad ...(4)$$

Eliminating $\frac{d\theta}{dt}$ between (3) and (4), we get

$$\frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2+r^2\cdot\frac{n\mu c}{r^4}\right]=\frac{\mu}{r}+\frac{\mu}{2c}(n-2).$$

Multiply throughout by $2r^2$.

$$\therefore \quad r^2\left(\frac{dr}{dt}\right)^2=2\mu r+\frac{\mu}{c}(n-2)\,r^2-n\mu c$$

$$=\frac{\mu}{c}\,[(n-2)\,r^2+2cr-nc^2]$$

$$=\frac{\mu}{c}\,[r-c]\,[(n-2)\,r+nc]$$

$$=\frac{\mu}{c}\,[r-c]\,[nc-(2-n)\,r]. \qquad \textit{Proved.}$$

CHAPTER II

THE INVERSE SQUARE LAW

(Planetary Motion)

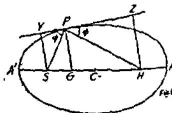
§ 1. The most important case of central orbits is that in which the force is an **attraction varying inversely as the square of the distance** from the centre of force. The law is said to be **Newtonian Law of Attraction** discovered by Newton which asserts that any two particles of matter whose masses are m and M attract one another when at a distance r with a force $\frac{\gamma mM}{r^2}$ where γ is a universal constant or constant of gravitation depending on the units of mass and length employed. The title planetary motion is due to the fact that the above law is found to hold good in the case of the motion of all planets is our solar system *i.e.* the earth moving round the sun, the motion of moon round the earth. Thus under the law $\frac{\gamma mM}{r^2}$ we shall study the motion of m under the attraction of M, if M is held fixed or if the mass M be so large that the effect on its motion of the attraction of m is negligible.

§ 2. *A particle moves in a path so that its acceleration is always directed to a fixed point and is equal to* $\frac{\mu}{(distance)^2}$.

Show that the path is a conic section and distinguish between the three cases that arise.

(Cal. Hon's. 62 ; Sagar 62 ; Agra 56, 47)

Here $P=\frac{\mu}{r^2}$. Hence the differential equation of the



1. The sum of the focal distances of a point on an ellipse is $2a$ = length of its major axis.

i.e. $SP+HP=2a,$

or $a+ex+a-ex=2a.$

Hence if $SP=r$, then $HP=2a-r$.

2 The product of perpendiculars drawn from the foci on any tangent to an ellipse is constant and equal to square of the semi-minor axis of the ellipse, *i.e.* $SY.HZ=b^2$.

Also the feet of the perpendiculars lie on the auxiliary circle, *i e.* the points Y and Z lie on $x^2+y^2=a^2$.

3. Also CZ is parallel to SP in the above figure.

4. The length of latus rectum $=\frac{2b^2}{a}$ where $b^2=a^2(1-e^2)$ or $a^2-b^2=a^2e^2$, e being the eccentricity of the ellipse.

5. $CH=CS=ae$.

6. The tangent and normal at any point are each equally inclined to focal radii of that point,

i.e. $\angle YPS=\angle ZPH$ or $\angle GPS=\angle GPH$.

7. The distance of the extremity of minor axis B, *i.e.* $(0, b)$ from the focus $(ae, 0)$ is a,

i.e. $HB^2=(ae-0)^2+(0-b)^2=a^2e^2+b^2$

or $HB^2=a^2e^2+a^2-a^2e^2=a^2$; $\therefore$ $HB=a$;

$\therefore$ $SB=2a-a=a$.

Exercise

Ex. 1. *Show that the velocity of a particle moving in an ellipse about a centre of force in the focus is compounded of two constant velocities μ/h perpendicular to radius vector and $\frac{e\mu}{h}$ perpendicular to major axis.*

(Delhi Hons. 55; Cal. Hons. 60; Agra 43, 62)

Hence deduce that the radial velocity is given by the equation

$$r^2\left(\frac{dr}{dt}\right)^2=\frac{\mu}{a}\{a(1+e)-r\}\{r-a(1-e)\}.$$

(Pb. 56, 65 ; Delhi Hons. 55, 61)

Also show that its velocity away from the sun is greatest when the radius vector to the planet is at right angles to the major axis of the path (i.e. when it is at the end of the latus rectum of the path) and that it then is $\dfrac{2\pi ae}{T\sqrt{(1-e^2)}}$ *where* $2a$ *is the major axis,* e *the eccentricity and* T *the periodic time.*

(Punjab 59; Cal. Hons. 60; Sagar 63; Agra 49, 52. 65)

Let the focus S be the centre of force and H be the empty focus. If v be the velocity, then we know that

$$vp=h \quad \text{or} \quad v=h/p$$

where p is perpendicular from centre of force on the tangent.

Fig 7

$$\therefore \quad v=\frac{h}{SY}=\frac{h}{b^2}\,HZ, \qquad \because \quad SY.HZ=b^2 \quad (\S\ 5{\cdot}2).$$

From above we find that v is proportional and perpendicular to HZ (HZ being $\perp$ to tangent *i e.* the direction of velocity).

From $\triangle$ CHZ we have, $\vec{HC}+\vec{CZ}=\vec{HZ}$

or $$\frac{h}{b^2}\vec{HC}+\frac{h}{b^2}\,\vec{CZ}=\frac{h}{b^2}\,\vec{HZ}.$$

Hence we can say that the velocity is equivalent to two velocities, *i e.* $\frac{h}{b^2}HC$ perpendicular to HC and $\frac{h}{b^2}\,.\,CZ$ perpendicular to CZ. But $HC=ae$ and $CZ=a$ (Z lies on the auxiliary circle $x^2+y^2=a^2$, $\therefore$ $CZ+\text{radius}=a$). Also we know that CZ is parallel to SP.

Hence the components of the velocities are

$$\frac{h}{b^2}\,.\,HC=\frac{h}{b^2}\,.\,ae=\frac{he}{b^2/a}=\frac{he}{l}=\frac{he}{h^2/\mu}=\frac{\mu e}{h}, \qquad \because \quad h^2=\mu l$$

and this component is perpendicular to HC *i.e.* perpendicular to major axis.

The other component of velocity $=\frac{h}{b^2} CZ$ and it is perpendicular to CZ or we may say perpendicular to radius vector SP which we know is parallel to CZ by § 5·3. But $CZ=a$.

$$\therefore \quad \text{This component}=\frac{h}{b^2} \cdot a=\frac{h}{b^2/a}=\frac{h}{l}=\frac{h}{h^2/\mu}=\frac{\mu}{h}.$$

Another form of above.

A particle is describing an ellipse about a centre of force at the focus Show that its velocity can be resolved into two components of constant magnitude, one perpendicular to the radius vector and the other perpendicular to major axis.

(Agra 62)

2nd Part. Again velocity away from the sun *i.e.* along SP is to be determined. It is equivalent to sum of the resolved parts of components found above along SP. The component $\mu/h \perp$ to SP will have no resolved part along SP, whereas the component $e\frac{\mu}{h}$ perpendicular to major axis will have $e\frac{\mu}{h}\cos\theta$ as its resolved part along SP.

Thus the velocity V away from the sun is

$$V=e\,\frac{\mu}{h}\cos\theta.$$

This is max. when $\cos\theta=1$ *i.e.* $\theta=0$ and in that case SP is perpendicular to major axis *i.e.* P is at the end of latus rectum of the path.

$$\therefore \quad \text{Max. value of } V=e\frac{\mu}{h}=\sqrt{\left(\frac{e^2\mu^2}{h^2}\right)}=\sqrt{\left(\frac{e^2\mu^2}{\mu l}\right)}$$

$$=\sqrt{\left(\frac{e^2\mu^2 \cdot a}{\mu \cdot b^2}\right)}=e\sqrt{\left(\frac{\mu a}{a^2(1-e^2)}\right)}$$

$$=e\sqrt{\left(\frac{\mu}{a(1-e^2)}\right)}. \qquad \ldots(1)$$

Also the periodic time $T=\frac{2\pi}{\sqrt{\mu}}a^{3/2}$ or $\sqrt{\mu}=\frac{2\pi}{T}a^{3/2}$.

Putting for $\sqrt{\mu}$ in (1), we get

the max. value of $V=e.\frac{2\pi}{T}a^{3/2}.\frac{1}{\sqrt{\{a(1-e^2)\}}}$

$$=\frac{2\pi ae}{T\sqrt{(1-e^2)}}.$$

Radial Velocity.

As discussed above radial velocity *i.e.* velocity along SP is

$$e\frac{\mu}{h}\cos\theta=\frac{e\mu}{h}\cos(90-t)=\frac{e\mu}{h}\sin t$$

where t is the angle which SP makes with the radius vector.

$$\therefore\quad \frac{dr}{dt}=\frac{e\mu}{h}\sin t$$

or
$$\left(\frac{dr}{dt}\right)^2=\frac{e^2\mu^2}{h^2}(1-\cos^2 t).$$

Now
$$h^2=\mu l \text{ and } \frac{l}{r}=1+e\cos t$$

and
$$l=\frac{b^2}{a}=a(1-e^2).$$

$$\therefore\quad \left(\frac{dr}{dt}\right)^2=\frac{e^2\mu^2}{\mu l}\left[1-\left(\frac{l-r}{er}\right)^2\right]$$

or
$$r^2\left(\frac{dr}{dt}\right)^2=\frac{\mu}{l}[r^2e^2-l^2-r^2+2lr]$$

$$=\mu\left[-l+2r-r^2\frac{(1-e^2)}{l}\right]$$

$$=\mu\left[-a(1-e^2)+2r-\frac{r^2(1-e^2)}{a(1-e^2)}\right]$$

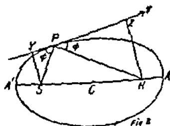
$$=\frac{\mu}{a}[-a^2(1-e^2)+2ra-r^2]$$

$$-\frac{\mu}{a}[a(1+e)-r][r-a(1-e)]$$

by factorisation.

Independent proof of 2nd part.

We know that tangent at any point on an ellipse is equally inclined to the focal radii of the point of contact



$\therefore \quad \angle SPY = \angle HPZ = \phi.$

$\therefore \quad \sin\phi = \frac{HZ}{PH} = \frac{SY}{PS}.$

$$\therefore \quad \sin^2\phi = \frac{HZ}{PH}\cdot\frac{SY}{PS} = \frac{b^2}{r(2a-r)}. \quad \text{...(1)}$$

$\therefore \quad SY.HZ = b^2.$ Also if $SP = r$, then $HP = 2a - r$ (§ 5).

Again the velocity away from the sun is the resolved part along SP of the velocity $v = v\cos\phi = V$.

Also the velocity in the case of an ellipse is given by

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right).$$

$$\therefore \quad V^2 = v^2\cos^2\phi = v^2(1-\sin^2\phi)$$

$$= \mu\left(\frac{2}{r}-\frac{1}{a}\right)\left(1-\frac{b^2}{r(2a-r)}\right) \quad \text{[from (1)]}$$

$$= \mu\left(\frac{2a-r}{ar}\right)\left[\frac{2ar-r^2-b^2}{r(2a-r)}\right] = \frac{\mu}{a}\left[\frac{2ar-r^2-b^2}{r^2}\right]$$

$$= \frac{\mu}{a}\left[\frac{2a}{r}-1-\frac{b^2}{r^2}\right]. \quad \text{...(2)}$$

Now V^2 is max. according as $\frac{2a}{r}-1-\frac{b^2}{r^2}$ is max.

Let $z = \frac{2a}{r} - 1 - \frac{b^2}{r^2}$. For max. value of z, $\frac{dz}{dr} = 0$,

$$\frac{dz}{dr} = -\frac{2a}{r^2} + \frac{2b^2}{r^3} = 0; \quad \therefore \quad r = \frac{b^2}{a}. \quad \text{...(3)}$$

Also $\frac{d^2z}{dr^2}=\frac{4a}{r^3}-\frac{6b^2}{r^4}=\frac{2}{r^3}\left[2a-\frac{3b^2}{r}\right]$

$$=2\ \frac{a^3}{b^6}\ [2a-3a],\ i.e.\ \text{-ive when } r=\frac{b^2}{a}.$$

$\therefore$ z is Max., when $r=\frac{b^2}{a}$=semi-latus rectum,

i.e. SP is semi-latus rectum, which in other words mean that the radius vector to the planet is perpendicular to major axis.

Putting $r=\frac{b^2}{a}$ in $z=\frac{2a}{r}-1-\frac{b^2}{r^2}$, we get

$$z=\frac{2a^2}{b^2}-1-\frac{b^2.a^2}{b^4}=\frac{a^2-b^2}{b^2}=\frac{a^2e^2}{a^2(1-a^2)}.$$

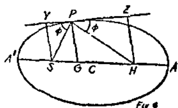
$$\therefore\quad V^2=\frac{\mu}{a}.\frac{e^2}{(1-e^2)}\text{ by (2).} \qquad ...(4)$$

Also if T be the periodic time, then

$$T=\frac{2\pi}{\sqrt{\mu}}\ a^{3/2}\quad\text{or}\quad \mu=\frac{4\pi^2a^3}{T^2}.$$

$$\therefore\quad V^2=\frac{4\pi^2a^3}{aT^2}.\frac{e^2}{(1-e^2)}\quad\text{or}\quad V=\frac{2\pi ae}{T\sqrt{(1-e^2)}}.$$

Ex. 2. *A particle describes an ellipse about a centre of force at the focus. Show that at any point of its path the angular velocity about the other focus varies inversely as the square of the normal at that point or varies as square of perpendicular from centre C to the tangent at P or varies inversely as square of CD where CD is a diameter conjugate to CP.*



We know that in a central orbit,

$$vp=h=r^2\frac{d\theta}{dt}=SP^2.\frac{d\theta}{dt},$$

where $\frac{d\theta}{dt}$ stands for the

angular velocity about the focus S. If ω be the angular velocity about the focus H, then by the above rule,

$$vp=h=HP^2 . \omega$$

or
$$v . HZ = HP^2 . \omega.$$

But
$$v=\frac{h}{p}=\frac{h}{SY}.$$

$$\therefore \quad \frac{h}{SY} . HZ = HP^2 \omega \quad \text{or} \quad \omega=\frac{h}{SY} . \frac{HZ}{HP^2}$$

or
$$\omega=\frac{h}{SY . HP} . \frac{HZ}{HP} \qquad \ldots(1)$$

Now in $\triangle$s SPY and HPZ, we have $\angle SPY = \angle HPZ$ and $\angle SYP = \angle HZP = \frac{\pi}{2}$, *i.e.* the triangles are similar

$$\therefore \quad \frac{HZ}{HP}=\frac{SY}{SP}. \qquad \ldots(2)$$

Hence from (1) and (2), we get

$$\omega=\frac{h}{SY . HP} . \frac{SY}{SP}=\frac{h}{SP . HP}=\frac{h}{(a+ex)(a-ex)}$$

or
$$\omega=\frac{h}{a^2-e^2x^2}=\frac{h}{a^2-e^2 . a^2 \cos^2 t}$$

if the point be taken as $(a \cos t, b \sin t)$

or
$$\omega=\frac{h}{a^2-(a^2-b^2)\cos^2 t}=\frac{h}{a^2 \sin^2 t+b^2 \cos^2 t} \qquad \ldots(3)$$

Again normal at P is $\frac{ax}{\cos t}-\frac{by}{\sin t}=a^2-b^2$.

It meets the major axis $y=0$ at G whose co-ordinates are $\left(\frac{a^2-b^2}{a} \cos t, 0\right)$ and P is $(a \cos t, b \sin t)$.

$\therefore$ The square of the length of normal

$$=PG^2=\left\{a \cos t-\frac{a^2-b^2}{a} \cos t\right\}^2+b^2 \sin^2 t$$

$$=\frac{b^4}{a^2}\cos^2 t+b^2 \sin^2 t=\frac{b^2}{a^2}(b^2\cos^2 t+a^2\sin^2 t) \qquad ...(4)$$

or $$PG^2=\frac{b^2}{a^2}\cdot\frac{h}{\omega} \text{ [by (3) and (4)]}$$

or $$\omega=\frac{b^2}{a^2}h\cdot\frac{1}{PG^2}.$$

Above shows that the angular velocity about the other focus H varies inversely as the square of the normal at that point.

Again tangent at P is $\frac{x}{a}\cos t+\frac{y}{b}\sin t-1=0$; $\therefore$ if p be perpendicular from cetre C on it then

$$p=\frac{1}{\sqrt{\left(\frac{\cos^2 t}{a^2}+\frac{\sin^2 t}{b^2}\right)}}=\frac{ab}{\sqrt{(b^2\cos^2 t+a^2\sin^2 t)}}.$$

$\therefore\quad b^2\cos^2 t+a^2\sin^2 t=\frac{a^2b^2}{p^2}$ or $\frac{h}{\omega}=\frac{a^2b^2}{p^2}$ or $\omega=\frac{h.p^2}{a^2b^2}$ by (3).

Again D the extremity of diameter conjugate to CP will be $\qquad(-a\sin t,\ b\cos t)$.

$\therefore\quad CD^2=a^2\sin^2 t+b^2\cos t=h/\omega$ by (3).

$\therefore\quad \omega=h/CD^2.$

Hence ω varies inversely as square of CD.

Ex. 3. *A particle is projected from the earth's surface with velocity v Show that if the diminution of gravity is taken into account, but the resistance of air neglected, the path is an ellipse of major axis* $\frac{2ga^2}{2ga-v^2}$, *where a is the earth's radius.*

We know that if the orbit be an ellipse, then

$$v^2=\mu\left(\frac{2}{r}-\frac{1}{a_1}\right),$$

where a_1 is semi-major axis of the ellipse.

Now for a particle on the earth's surface $r=a$. Also for any particle on the surface of the earth acceleration is g.

$$\therefore \frac{\mu}{r^2}=g \quad \text{or} \quad \frac{\mu}{a^2}=g. \quad \because \quad r=a, \quad \therefore \quad \mu=ga^2.$$

$$\therefore \quad v^2=ga^2\left[\frac{2}{a}-\frac{1}{a_1}\right] \quad \text{or} \quad 2ga-v^2=\frac{ga^2}{a_1}.$$

$$\therefore \quad a_1=\frac{ga^2}{2ga-v^2}$$

Hence major axis $=2a_1=\dfrac{2ga^2}{2ga-v^2}$. Proved.

Ex. 4. (a) *If v_1 and v_2 are the linear velocities of a planet when it is respectively nearest and farthest from the sun, prove that* $(1-e)\, v_1=(1+e)\, v_2$. **(Rajputana 62)**

Taking S the focus as sun, the nearest position is A', where
$SA'=a-ae=a\,(1-e)$.
and farthest position is
$SA=a+ae=a\,(1+e)$.

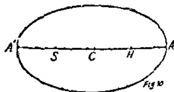


Fig 10

Also $v^2=\mu\left(\dfrac{2}{r}-\dfrac{1}{a}\right)$.

$$\therefore \quad v_1^2=\mu\left[\frac{2}{a(1-e)}-\frac{1}{a}\right]=\frac{\mu}{a}\,\frac{1+e}{1-e},$$

$$v_2^2=\mu\left[\frac{2}{a(1+e)}-\frac{1}{a}\right]=\frac{\mu}{a}\,\frac{1-e}{1+e}.$$

$$\therefore \quad \frac{v_1^2}{v_2^2}=\frac{1+e}{1-e}\div\frac{1-e}{1+e}=\frac{(1+e)^2}{(1-e)^2}.$$

$$\therefore \quad v_1\,(1-e)=v_2\,(1+e). \qquad \text{Proved.}$$

Ex. 4. (b) *If ω be the angular velocity of a planet at the nearer end of the major axis, prove that its period is*

$$\frac{2\pi}{\omega}\sqrt{\left\{\frac{1+e}{(1-e)^3}\right\}}.$$

At the nearer end A', $r=SA'=a-ae=a(1-e)$.

Also $\quad r^2\dfrac{d\theta}{dt}=h=\sqrt{(\mu.l)}$ [§ 3 P. 111]

or $\quad a^2(1-e)^2\omega=\sqrt{\left(\mu.\dfrac{b^2}{a}\right)}=\sqrt{\{\mu a(1-e^2)\}}$.

$$\therefore \quad \sqrt{\mu}=\frac{a^2(1-e)^2\omega}{\sqrt{\{a(1-e)(1+e)\}}}$$

or
$$\sqrt{\mu}=a^{3/2}\frac{(1-e)^{3/2}}{\sqrt{(1+e)}}\omega \qquad ...(1)$$

If T be period, then

$$T=\frac{2\pi}{\sqrt{\mu}}a^{3/2}=\frac{2\pi}{\omega}\sqrt{\left\{\frac{1+e}{(1-e)^3}\right\}}\text{ by (1)}.$$

Ex. 5. *A particle describes an ellipse under a force $\dfrac{\mu}{(distance)^2}$ towards the focus. If it was projected with velocity V from a point distant r from the centre of force, show that its periodic time is*

$$\frac{2\pi}{\sqrt{\mu}}\left[\frac{2}{r}-\frac{V^2}{\mu}\right]^{-3/2}. \qquad \text{(Vikram 63)}$$

$$V^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right);\quad \therefore\quad \frac{1}{a}=\left(\frac{2}{r}-\frac{V^2}{\mu}\right).$$

Also $\quad T=\dfrac{2\pi}{\sqrt{\mu}}.a^{3/2}=\dfrac{2\pi}{\sqrt{\mu}}\left(\dfrac{2}{r}-\dfrac{V^2}{\mu}\right)^{-3/2}$.

Ex. 6. *Show that the velocity of a planet at any point of its orbit is the same as it would have been if it had fallen to that point from rest at a distance from the sun equal to the length of the major axis.*

Velocity at any point is $v^2=\mu\left(\dfrac{2}{r}-\dfrac{1}{a}\right)$. $\qquad ...(1)$

Again if the particle were to fall straight towards the sun from a distance r to a distance $2a$, measured from sun,

then the equation of motion would be

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2}$$

or $\quad v\frac{dv}{dx}=-\frac{\mu}{x^2}\quad$ or $\quad v\,dv=-\frac{\mu}{x^2}\,dx.$

Integrating, $\qquad v^2=\frac{2\mu}{x}+A.$

When $x=2a$, $v=0$; $\therefore$ $A=-\frac{\mu}{a}$.

$$\therefore\quad v^2=\frac{2\mu}{x}-\frac{\mu}{a}.$$

If V be the velocity when $x=r$, then

$$V^2=\frac{2\mu}{r}-\frac{\mu}{a}=\mu\left(\frac{2}{r}-\frac{1}{a}\right)=v^2.$$

Hence proved.

Ex. 7. *If the velocity of a body in an elliptic orbit, major axis 2a, is the same at a certain point P, whether the orbit is being described in a periodic time T about one focus or in periodic time T′ about the other focus S′, prove that*

$$SP=\frac{2aT'}{T+T'}\ \textit{and}\ S'P=\frac{2aT}{T+T'}.$$

If $SP=r$, then $S'P=2a-r$.

Taking S as the focus and S' as empty focus, we have

$$v^2=\mu\left[\frac{2}{r}-\frac{1}{a}\right]\quad\text{and}\quad T=\frac{2\pi}{\sqrt{\mu}}a^{3/2}. \qquad ...(1)$$

Again taking S' as focus and S as empty focus, $S'P=2a-r$, we have

$$V^2=\mu'\left[\frac{2}{2a-r}-\frac{1}{a}\right]\quad\text{and}\ T'=\frac{2\pi}{\sqrt{\mu'}}a^{3/2} \qquad ...(2)$$

Since velocities are given to be equal *i.e.* $v^2=V^2$,

$$\therefore\ \mu\left[\frac{2}{r}-\frac{1}{a}\right]=\mu'\left[\frac{2}{2a-r}-\frac{1}{a}\right]. \qquad ..(3)$$

Now $\mu=\frac{4\pi^2}{T^2}a^3$ and $\mu'=\frac{4\pi^2}{T'^2}a^3$.

Putting for μ and μ' in (3), we get

$$\frac{4\pi^2}{T^2}a^3\left[\frac{2}{r}-\frac{1}{a}\right]=\frac{4\pi^2}{T'^2}a^3\left[\frac{2}{2a-r}-\frac{1}{a}\right]$$

or $$\frac{1}{T^2}\frac{2a-r}{ar}=\frac{1}{T'^2}\frac{r}{a(2a-r)}$$

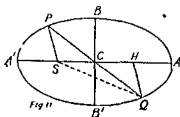
or $$T'^2(2a-r)^2=T^2.r^2$$

or $$T'(2a-r)=Tr \quad\text{or}\quad 2a.T'=r(T+T').$$

$$\therefore\quad SP=r=\frac{2aT'}{T+T'}.$$

$$\therefore\quad S'P=2a-r=2a-\frac{2aT'}{T+T'}=\frac{2aT}{T+T'}.$$ **Proved.**

Ex. 8. *A particle describes an ellipse as a central orbit about the focus. Prove that the velocity at the end of the minor axis is geometric mean between the velocities at the ends of any diameter.*



B is the end of minor axis.

$\therefore\quad SB=\sqrt{(a^2e^2+b^2)}$

$=\sqrt{a^2}=a.$ (§ 5·7) ...(1)

Also in $\triangle$s CSP and CHQ, we have

$CP=CQ$, as PQ is a diameter and C is centre

$CS=CH$; $\therefore\quad SP=HQ$.

$\therefore\quad SP+SQ=HQ+SQ=2a$ [as sum of the focal distances$=2a$]

$$\therefore\quad SP+SQ=2a. \qquad ...(2)$$

Let V be the velocity at the end of minor axis *i.e.* at B where $SB=a$; then by $v^2=\mu\left[\frac{2}{r}-\frac{1}{a}\right]$, we have

$$V^2 = \mu \left[\frac{2}{a} - \frac{1}{a}\right] = \frac{\mu}{a}. \quad ...(3)$$

Again if v_1 and v_2 be the velocities at P and Q, then

$$v_1^2 = \mu \left[\frac{2}{SP} - \frac{1}{a}\right], \; v_2^2 = \mu \left[\frac{2}{SQ} - \frac{1}{a}\right].$$

$$\therefore \quad v_1^2 v_2^2 = \mu^2 \left[\frac{4}{SP.SQ} - \frac{2}{a} \cdot \frac{SP+SQ}{SP.SQ} + \frac{1}{a^2}\right]$$

or $$v_1^2 v_2^2 = \mu^2 \left[\frac{4}{SP.SQ} - \frac{2}{a} \cdot \frac{2a}{SP.SQ} + \frac{1}{a^2}\right] \text{ by (2)}$$

$$= \frac{\mu^2}{a^2} = V^2 \text{ by (3)}.$$

V is geometric mean of v_1 and v_2.

Ex. 9. *A particle describes an ellipse under a force to the focus S. When the particle is at one extremity of the minor axis, its kinetic energy is doubled without any change in the direction of motion. Prove that the particle proceeds to describe a parabola*

From Q. 8 if V be the velocity at the end of minor axis *i.e.* $SB = a$, then $V^2 = \mu \left[\frac{2}{a} - \frac{1}{a}\right] = \frac{\mu}{a}$. .. (1)

Now when K. E. is doubled, let the velocity be v; then $\frac{1}{2}mv^2 = 2 \cdot \frac{1}{2}m \cdot V^2$ by the given condition

$$\therefore \quad v^2 = 2V^2 \quad \text{or} \quad v^2 = 2 \cdot \frac{\mu}{a} \text{ by (1)}$$

But we know from § 2 that in the case of parabolic orbit $v^2 = \frac{2\mu}{r} = \frac{2\mu}{a}$, $\because$ $r = a$. Hence we conclude that the particle proceeds to describe a parabola.

Ex. 10. *If a planet were suddenly stopped in its orbit supposed circular, show that it would fall into the sun in a time which is* $\frac{\sqrt{2}}{8}$ *times the period of the planet's revolution.*

(Agra 1953, '60)



Let the particle moving in a circular orbit about S be suddenly stopped at P, so that its velocity is reduced to zero and now it begins to move towards S in a straight line PS. The equation of motion is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2} \quad (x \text{ decreasing})$$

or $$v\frac{dv}{dx}=-\frac{\mu}{x^2} \quad \text{or} \quad v\,dv=-\frac{\mu}{x^2}\,dx.$$

Integrating, $v^2=\dfrac{2\mu}{x}+A.$

When $x=a$ i.e. planet is at P, then $v=0$.

$$\therefore \quad A=-\frac{2\mu}{a}.$$

$$\therefore \quad v^2=2\mu\left(\frac{1}{x}-\frac{1}{a}\right)=2\mu\left(\frac{a-x}{ax}\right).$$

$$\therefore \quad v=\frac{dx}{dt}=-\sqrt{(2\mu)}\sqrt{\left(\frac{a-x}{ax}\right)} \quad (x \text{ decreasing}).$$

$$\therefore \quad \int_a^0 -\sqrt{\left(\frac{ax}{a-x}\right)}\,dx=\int_{t=0}^{t}\sqrt{(2\mu)}\,dt. \qquad \ldots(1)$$

Limits are chosen because at $t=0$, $x=a=SP$ when $t=t$ say, $x=0$, *i.e.* planet is at S.

Putting $x=a\cos^2\theta$, $\therefore$ $dx=-2a\cos\theta\sin\theta\,d\theta$ in (1),

$$\int_0^{\pi/2}\sqrt{a}.\frac{\cos\theta}{\sin\theta}\,2a\cos\theta\sin\theta\,d\theta=\sqrt{(2\mu)}\,t$$

or $$2a^{3/2}\int_0^{\pi/2}\cos^2\theta\,d\theta=\sqrt{(2\mu)}\,t\,; \quad \therefore \quad t=\frac{2a^{3/2}}{\sqrt{(2\mu)}}.\frac{1}{2}.\frac{\pi}{2}. \qquad \ldots(2)$$

Also if T be the periodic time of planet's revolution, then

$$T=\frac{2\pi}{\sqrt{\mu}}\,a^{3/2}. \qquad \ldots(3)$$

$\therefore \quad t=\frac{1}{4\sqrt{2}}T \quad \text{or} \quad t=\frac{\sqrt{2}}{8}T.$ Proved.

Ex. 11. *If the velocity of the earth at any point of its orbit assumed to be circular were increased by about one half, prove that it would describe a parabola about the sun as focus.*

Refer § 3 Cor. 2 P 105. If V be the velocity for the description of a circle, then

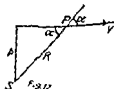
$$\frac{V^2}{r}=\text{normal acceleration}=\frac{\mu}{r^2}\,; \quad \therefore \quad V^2=\frac{\mu}{r}. \qquad \ldots(1)$$

Also if v be the velocity for a parabolic path, then

$$v^2=\frac{2\mu}{r}. \qquad \ldots(2)$$

$\frac{v^2}{V^2}=2$ or $v=\sqrt{2}\,.\,V=(1\cdot41)\,V=(1\tfrac{1}{2})\,V$ nearly.

Ex. 12. *A particle moves with a central acceleration* $\frac{\mu}{(\text{distance})^2}$*. It is projected with velocity* V *at a distance* R*. Show that its path is a rectangular hyperbola if the angle of projection is* $\sin^{-1}\frac{\mu}{VR\left(V^2-\frac{2\mu}{R}\right)^{1/2}}$. (Cal. Hon's. '63; Agra '47)



We know that in a central orbit $v=\frac{h}{p}$ and from the figure $p=R\sin\alpha$ and $v=V$ given.

Also $h=\sqrt{(\mu l)}=\sqrt{\left(\mu\,.\,\frac{b^2}{a}\right)}$

$$=\sqrt{\left(\mu\,.\,\frac{a^2}{a}\right)}=\sqrt{(\mu a)}$$

because in a rectangular hyperbola, $b=a$.

$$\therefore \quad V=\frac{\sqrt{(\mu a)}}{R\sin\alpha} \quad or \quad \sin\alpha=\frac{\sqrt{(\mu a)}}{VR}. \qquad \ldots(1)$$

In the case of a hyperbolic orbit $v^2=\mu\left(\frac{2}{r}+\frac{1}{a}\right)$ [§ 3].

When $r=R$, $v=V$ given.

$$\therefore\quad V^2=\mu\left[\frac{2}{R}+\frac{1}{a}\right];\quad \therefore\quad V^2-\frac{2\mu}{R}=\frac{\mu}{a}$$

or
$$\sqrt{a}=\frac{\sqrt{\mu}}{\left(V^2-\frac{2\mu}{R}\right)^{1/2}}.$$

Putting for $\sqrt{a}$ from (2) in (1), we get

$$\sin\alpha=\frac{\sqrt{\mu}}{VR}\cdot\frac{\sqrt{\mu}}{\left(V^2-\frac{2\mu}{R}\right)^{1/2}}=\frac{\mu}{VR\left(V^2-\frac{2\mu}{R}\right)^{1/2}}.$$

(b) *Prove that if when the particle is at a distance r from the focus its velocity is v in a direction making an angle ϕ with the radius vector, then*

$$e^2\mu^2=(v^2r-\mu)^2\sin^2\phi+\mu^2\cos^2\phi.$$

$$v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right). \qquad \text{...(1)}$$

Also $vp=h$ when $p=r\sin\phi$.

$$\therefore\quad vr\sin\phi=h. \qquad \text{.. (2)}$$

Also
$$h^2=\mu l=\mu\,\frac{b^2}{a}=\mu a\,(1-e^2). \qquad \text{...(3)}$$

In the relations to be proved we do not require a and h and hence we have to eliminate a and h between (1), (2) and (3).

$$\therefore\quad v^2r^2\sin^2\phi=\mu a\,(1-e^2)\quad \text{from (2) and (3).}$$

$$\therefore\quad \frac{1}{a}=\frac{\mu\,(1-e^2)}{v^2r^2\sin^2\phi}.$$

Putting the value of $\frac{1}{a}$ in (1), we get

$$v^2=\frac{2\mu}{r}-\mu\cdot\frac{\mu\,(1-e^2)}{v^2r^2\sin^2\phi}.$$

or $$\left(\frac{v^2 r-2\mu}{r}\right) v^2 r^2 \sin^2\phi=-\mu^2+\mu^2 e^2.$$

$$\therefore\quad e^2\mu^2=(v^4 r^2-2\mu v^2 r)\sin^2\phi+\mu^2(\cos^2\phi+\sin^2\phi)$$
$$=(v^4 r^2-2\mu v^2 r+\mu^2)\sin^2\phi+\mu^2\cos^2\phi$$
$$=(v^2 r-\mu)^2\sin^2\phi+\mu^2\cos^2\phi.$$

(c) *A particle is projected from a point P with velocity V in a direction making an angle α with the radius vector. If V were V′, the orbit would be a parabola. Prove that if the orbit is hyperbolic, the angle between the asymptotes is*

$$2\tan^{-1}\left[\frac{2V}{V'}\left(\frac{V^2}{V'^2}-1\right)^{1/2}\right]\sin\alpha.$$

$$V'^2=\mu\frac{2}{r} \qquad \text{for a parabola,}$$

$$V^2=\mu\left(\frac{2}{r}+\frac{1}{a}\right) \qquad \text{for a hyperbola.}$$

Angle between the asymptotes of a hyperbola

$$=2\tan^{-1}\frac{b}{a}$$

$$=2\tan^{-1}\sqrt{\left(\frac{b^2}{a^2}\right)}=2\tan^{-1}\sqrt{\left(\frac{l}{a}\right)} \quad \because\ l=\frac{b^2}{a}$$

$$=2\tan^{-1}\sqrt{\left(\frac{\mu l}{a\mu}\right)}=2\tan^{-1}\sqrt{\left(\frac{h^2}{\mu a}\right)} \quad \because\ h^2=\mu l. \qquad \ldots(1)$$

Now $$\frac{h^2}{a\mu}=\frac{v^2p^2}{a\mu} \quad \because\ h=vp$$

$$=\frac{\mu\left(\frac{2}{r}+\frac{1}{a}\right)}{a\mu}\cdot r^2\sin^2\alpha \quad \because\ p=r\sin\alpha$$

$$=\left(\frac{2}{r}+\frac{1}{a}\right)\frac{r^2\sin^2\alpha}{a}=\left(2\frac{r}{a}+\frac{r^2}{a^2}\right)\sin^2\alpha$$

or $$\frac{h^2}{a\mu}=\frac{r}{a}\left(2+\frac{r}{a}\right)\sin^2\alpha \qquad \ldots(2)$$

Again $V^2=\mu \cdot \frac{2}{r}+\frac{\mu}{a}=V'^2+\frac{\mu}{a}$

or $$V^2=V'^2+\frac{V'^2 . r}{2 . a} \quad \because \quad V'^2=\frac{2\mu}{r}.$$

$$\therefore \quad \frac{V^2}{V'^2}-1=\frac{r}{2a} \quad \text{or} \quad \frac{r}{a}=2\left(\frac{V^2}{V'^2}-1\right).$$

Putting for $\frac{r}{a}$ in (2), we get

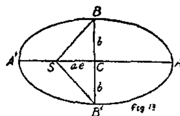
$$\frac{h^2}{a\mu}=2\left(\frac{V^2}{V'^2}-1\right)\left[2+2\frac{V^2}{V'^2}-2\right]\sin^2\alpha=4\frac{V^2}{V'^2}\left(\frac{V^2}{V'^2}-1\right)\sin^2\alpha.$$

$$\therefore \quad \text{Angle}=2\tan^{-1}\sqrt{\left(\frac{h^2}{a\mu}\right)} \text{ by (1)}$$

$$=2\tan^{-1}\left[2\frac{V}{V'}\left(\frac{V^2}{V'^2}-1\right)^{1/2} . \sin\alpha\right].$$

Ex. 13. *Prove that the time taken by the earth to travel over half of its orbit remote from the sun separated by the minor axis is about 2 days more than half the year. The eccentricity of the orbit is* $\frac{1}{60}$.

$h=r^2\frac{d\theta}{dt}=2.$ (rate of description of area in a central orbit is constant).



If t be the time of describing the sectorial area $SB'ABS$ i.e. remote half from the sun $B'AB$, the rate of description of

$$\text{area}=\frac{\text{araa } SB'ABS}{t} \qquad ...(1)$$

Again the earth travels round the sun describing the ellipse in one year.

$$\therefore \quad \text{Rate of description of area}=\frac{\text{area of ellipse}}{\text{one year}}. \qquad ...(2)$$

Since rate of description is constant, hence from (1) and (2), we get

$$\frac{\text{Area } SB'ABS}{t}=\frac{\text{Area of ellipse}}{\text{one year}}. \quad ...(3)$$

Now area $SB'ABS$ = Area $ABB' + \triangle SBB'$

$$=\frac{\pi ab}{2}+\tfrac{1}{2}.2b.ae.$$

Also area of ellipse $=\pi ab$.

$$\therefore \text{ from (3), } t=\frac{\frac{\pi ab}{2}+abe}{\pi ab}.\text{one year}$$

$$=\left(\frac{1}{2}+\frac{e}{\pi}\right).\text{one year}$$

$$=\tfrac{1}{2}\text{ year}+\tfrac{1}{60}.\tfrac{7}{22}\times 365 \text{ days}$$

$$=\tfrac{1}{2}\text{ year}+2 \text{ days nearly.}$$ Hence proved.

Ex. 14. *A comet is moving in a parabola about the sun as focus, when at the end of its latus rectum its velocity suddenly becomes altered in the ratio $n : 1$, where $n < 1$. Show that the comet will describe an ellipse whose eccentricity is $\sqrt{(1-2n^2+2n^4)}$ and whose major axis is $\frac{l}{1-n^2}$ where $2l$ was the latus rectum of the parabolic path.*

(Punjab 58; Agra 46, 50, 54, 61)

In the case of parabola, $v^2=\frac{2\mu}{r}$

where $r=SL=l$; $\therefore$ $v^2=\frac{2\mu}{l}$. ...(1)

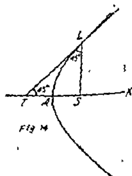


Fig 14

Now the velocity is altered to v', where

$$v'=\frac{n}{1}v=n\sqrt{\left(\frac{2\mu}{l}\right)}. \quad ...(2)$$

Now the comet describes an elliptic orbit whose major axis is say $2a'$; then we know that for an elliptic orbit

$$v^2=\mu\left[\frac{2}{r}-\frac{1}{a}\right].$$

$$\therefore \quad v'^2=\mu\left[\frac{2}{l}-\frac{1}{a'}\right]$$

at the end of L. R. where $r=l$ of parabola which the comet was originally describing and when $r=l$ the comet begins to describe an ellipse

or $\quad n^2\frac{2\mu}{l}=\mu\left[\frac{2}{l}-\frac{1}{a'}\right]$, by (2) ; $\therefore \frac{1}{a'}=\frac{2}{l}(1-n^2)$. ...(3)

$\therefore$ Major axis is $2a'=\dfrac{l}{1-n^2}$.

Now we have to find the eccentricity of elliptic orbit.

We know that $Vp=h=\sqrt{(\mu l')}=\sqrt{[\mu a'(1-e^2)]}$, where l' is the semi-latus rectum of elliptic orbit.

Also when V is v', then $p=l\sin 45°$, where p is $\perp$ from S on TL.

$$\therefore \quad v'.l\sin 45°=\sqrt{[\mu a'(1-e^2)]}.$$

Squaring and putting for μ' and a' from (2) and (3),

$$n^2.\frac{2\mu}{l}.l^2\sin^2 45°=\mu a'(1-e^2)$$

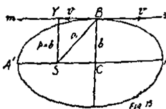
or $$2.n^2l.\frac{1}{2}=\frac{l}{2(1-n^2)}(1-e^2)$$

or $$2n^2(1-n^2)=1-e^2 \quad \text{or} \quad e^2=1-2n^2+2n^4$$

or $$\therefore e=\sqrt{(1-2n^2+2n^4)}.$$ Hence proved.

Ex. 15. *Two inelastic particles of masses 3m and m are describing the same ellipse of eccentricity e in opposite directions under a force to the focus. They collide and coalesce at an extremity of minor axes. Prove that the eccentricity of the new orbit is* $\frac{1}{4}\sqrt{(9+7e^2)}$.

We have already shown in Q. 8 that $SB=a$, *i.e.* $r=a$ at B, the end of minor axis.



Let the equal velocity of the two particles before they collide at B be v_1 in opposite direction.

$$\therefore \quad v_1^2=\mu\left[\frac{2}{r}-\frac{1}{a}\right]=\mu\left[\frac{2}{a}-\frac{1}{a}\right]=\frac{\mu}{a}. \qquad \ldots(1)$$

Suppose that after impact when they coalesce and form one particle of mass $3m+m=4m$, the velocity be V; then we know that momentum after = momentum before.

$$\therefore \quad 4mV=3mv_1-mv_1=2mv_1\ ; \quad \therefore \quad V=\frac{v_1}{2}$$

or
$$V^2=\frac{1}{4}v_1^2=\frac{1}{4}\frac{\mu}{a} \text{ by (1).} \qquad \ldots(2)$$

Now we know that when $V^2<\frac{2\mu}{r}$ *i.e.* $<\frac{2\mu}{a}$ at $r=a$, then the particle with velocity V will describe an ellipse. $\left[\text{Here } V^2=\frac{1}{4}\frac{\mu}{a} \text{ by (2)}\right]$. Let $2a'$ be the major axis of this new ellipse which the combined body will begin to trace from B and for an ellipse,

$$v^2=\mu\left[\frac{2}{r}-\frac{1}{a'}\right]. \text{ When } r=a,\ v^2=V^2=\frac{1}{4}\frac{\mu}{a}.$$

$$\therefore \quad \frac{1}{4}\frac{\mu}{a}=\mu\left[\frac{2}{a}-\frac{1}{a'}\right]; \quad \therefore \quad 7a'=4a \quad \text{or} \quad a'=\frac{4a}{7}.$$

Again $Vp=h=\sqrt{(\mu l')}=\sqrt{[\mu a'(1-e'^2)]}$, where e' is the eccentricity of the new ellipse. From the figure it is clear that $p=b$.

$$\therefore \quad \frac{1}{4}\frac{\mu}{a}.b^2=\mu a'(1-e'^2). \quad \text{Put} \quad a'=\frac{4a}{7}$$

or $$\frac{1}{4}\frac{b^2}{a}=\frac{4a}{7}(1-e'^2) \quad \text{or} \quad 1-e'^2=\frac{7}{16}\frac{b^2}{a^2}$$

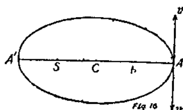
or $$e'^2=1-\frac{7}{16}\frac{a^2(1-e^2)}{a^2}=\frac{9+7e^2}{16}.$$

$$\therefore \quad e'=\tfrac{1}{4}\sqrt{(9+7e^2)}.$$

Ex. 16. *A planet of mass M and periodic time T when at its greatest distance from the sun comes into collision with a meteor of mass m, moving in the same orbit in the opposite direction with velocity v. If m/M be small, show that the major axis of the planet's path is reduced by*

$$\frac{4m}{M}\frac{vT}{\pi}\sqrt{\left(\frac{1-e}{1+e}\right)}.$$

(Agra 53, 55 ; Vikram 62 ; Delhi Hons. 63 ; Sagar 62)



The planet is at its greatest distance from the sun when at A where $SA=a+ae=r$.

After impact if v' be the common velocity, then

$(M+m)\,v'=Mv-mv=(M-m)\,v$.

$$\therefore \quad v'=\left(1-\frac{m}{M}\right)\left(1+\frac{m}{M}\right)^{-1}v=\left(1-\frac{m}{M}\right)\left(1-\frac{m}{M}+\ldots\right)v$$

or $$v'=\left(1-\frac{2m}{M}\right)v, \quad \because \ \frac{m}{M} \text{ is small.} \qquad \ldots(1)$$

Clearly v' is less than v and hence the new orbit described by the combined body will also be an ellipse and suppose that its major axis is $2a'$.

Since v is the velocity of original ellipse when $r=a(1+e)$, we have on using

$$v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right),$$

$$v^2=\mu\left[\frac{2}{a(1+e)}-\frac{1}{a}\right]=\frac{\mu}{a}\frac{1-e}{1+e}. \qquad \ldots(2)$$

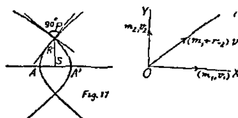
Ex. 37. *Two particles of masses m_1 and m_2 moving in coplanar parabolas round the sun collide at right angles and coalesce when their common distance from the sun is R. Show that the subsequent path of the combined particle is an ellipse of major axis* $\frac{(m_1+m_2)^2}{2m_1m_2}R$.

(Delhi Hons. 62, 64 ; Agra 47, 62 ; Punjab 56)

Let v_1 and v_2 be the velocities of the two particles when at P where $SP=R=r$ and we know that velocity in a parabolic orbit is given by $v^2=\frac{2\mu}{r}$.

$$\therefore \quad v_1^2=\frac{2\mu}{R}=v_2^2 \text{ for the point } P. \qquad \ldots(1)$$

After collision they form one body of mass (m_1+m_2) and since their velocities before impact were perpendicular, we have if the velocity of the combined particle be v, then momentum before impact are m_1v_1, m_2v_2 in perp. directions and momentum after$=(m_1+m_2)V$.



$$\therefore \quad (m_1+m_2)^2 . V^2=m_1^2v_1^2+m_2^2v_2^2=(m_1^2+m_2^2)\, v_1^2 \text{ [by (1)]}$$

$\because$ the resultant R of two forces P and Q acting at 90° is given by $R^2=P^2+Q^2$

or
$$V^2=\frac{m_1^2+m_2^2}{(m_1+m_2)^2}v_1^2. \qquad \ldots(2)$$

Since $m_1^2+m_2^2 < (m_1+m_2)^2$, hence $V^2 < v_1^2$, *i.e.* $< \frac{2\mu}{R}$.

parabolic orbit before impact, then $v^2=\frac{2\mu}{r}$. ...(1)

V the velocity of combined body is given by

$$(m+nm)\,V=mv\ ;\quad \therefore\quad V=\frac{v}{1+n}.$$

$$\therefore\quad V^2=\frac{1}{(n+1)^2}.v^2=\frac{1}{(n+1)^2}.\frac{2\mu}{r}\ \text{[by (1)]}. \qquad ...(2)$$

Clearly V^2 is less than $\frac{2\mu}{r}$ so that the particle will now describe an ellipse whose eccentricity may be taken as e and semi-major axis a'.

$$\therefore\quad V^2=\mu\left[\frac{2}{r}-\frac{1}{a'}\right]\quad \text{or}\quad \frac{2\mu}{r\,(n+1)^2}=\mu\left[\frac{2}{r}-\frac{1}{a'}\right].$$

$$\therefore\quad \frac{2}{r}\left[1-\frac{1}{(n+1)^2}\right]=\frac{1}{a'}\quad \text{or}\quad a'=\frac{r}{2}.\frac{(n+1)^2}{n\,(n+2)}. \qquad ...(3)$$

Also $\quad Vp=h=\sqrt{(\mu\,.\,l)}=\sqrt{[\mu\,.\,a'\,(1-e^2)]}$

when $\quad V^2=\frac{2\mu}{r}.\frac{1}{(n+1)^2}$ and in a parabola $p^2=ar$.

$$\therefore\quad \frac{2\mu}{r}.\frac{1}{(n+1)^2}.ar=\mu a'\,(1-e^2)$$

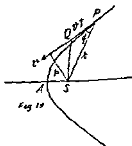
or $\quad 2\mu a.\frac{1}{(n+1)^2}=\mu\,(1-e^2).\frac{r}{2}\,\frac{(n+1)^2}{n\,(n+2)}$ by (3). Put for r.

$$\therefore\quad 4an\,(n+2)=(n+1)^4\,(1-e^2).\frac{a}{\cos^2\theta/2}. \qquad ...(4)$$

$$\therefore\quad (1-e^2)=\frac{4n\,(n+2)}{(n+1)^4}\cos^2\frac{\theta}{2}. \qquad \text{Proved.}$$

Ex. 19. *A particle is describing parabolic orbit (latus rectum 4a) about a centre of force (μ) in the focus and on its arriving at a distance r from the focus moving towards the vertex the centre of the force ceases to act for a certain time T. When the force begins again to operate, prove that the new orbit will be an ellipse, parabola or hyperbola according as*

$$T <=> 2r\sqrt{\left(\frac{r-a}{2\mu}\right)}. \qquad \text{(Agra 56)}$$



Let the particle be moving with velocity v moving towards the vertex and when at P the force ceases to act for time T and the particle arrives at Q where $\quad PQ=vT$,

$$v^2=\frac{2\mu}{r},\ PQ=vT$$

$$p=r\sin\phi \ \text{ or } \ \sin\phi=\frac{p}{r}=\frac{\sqrt{(ar)}}{r}=\sqrt{\frac{a}{r}}$$

as the pedal equation of a parabola is $p^2=ar$.

Now the particle has a velocity v at Q where $r=SQ$ and it will now describe an ellipse, parabola or hyperbola according as

$$v^2<=>\frac{2\mu}{SQ}$$

or $$\frac{2\mu}{r}<=>\frac{2\mu}{SQ} \ \text{ or } \ SQ^2<=>r^2. \qquad ...(1)$$

Now we have to find the value of SQ.

From triangle PSQ by cosine formula,

$$SQ^2=SP^2+PQ^2-2SP.PQ\cos\phi$$
$$=r^2+v^2T^2-2r.vT\sqrt{(1-\sin^2\phi)}$$
$$=r^2+v^2T^2-2r.vT\sqrt{\left(1-\frac{a}{r}\right)} \ \text{ as } \ \sin\phi=\sqrt{\frac{a}{r}}.$$

Hence from (1), we get the condition for ellipse, parabola or hyperbola as

$$r^2+v^2T^2-2vT\sqrt{(r^2-ar)}<=>r^2$$

or $$v^2T^2<=>2vT\sqrt{(r^2-ar)}$$

or $$T<=>\frac{2}{v}\sqrt{(r^2-ar)}$$

or $$T <=> 2\sqrt{\left(\frac{r}{2\mu}\right)}\sqrt{(r^2-ar)}, \quad \because \quad v^2=\frac{2\mu}{r}$$

or $$T <=> 2r\sqrt{\left(\frac{r-a}{2\mu}\right)}.$$

Ex. 20. *A body is moving in an ellipse about a centre of force in the focus; when it arrives at P, the direction of motion is turned through a right angle, the speed being unaltered. Prove that the body will describe an ellipse whose eccentricity varies as the distance of P from the centre.* **(Agra 58)**

The particle is moving with velocity v and at P the direction of velocity is turned through a right angle without change in magnitude. The velocity in an elliptic orbit is

Fig 20

$$v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right). \qquad \ldots(1)$$

After the direction is turned at P, v, μ and r remain unchanged and hence the path will be another ellipse; but its major axis will be same $2a$ because if $2a'$ be the new major axis, then

$$v^2=\mu\left(\frac{2}{r}-\frac{1}{a'}\right), \qquad \ldots(2)$$

and from (1) and (2), wet get $a=a'$.

Again let e' be the eccentricity of the new ellipse and p' the perpendicular on the tangent from focus S,

$$vp'=h=\sqrt{[\mu l]}=\sqrt{[\mu\,.\,a\,(1-e'^2)]}, \quad \because \quad l=\frac{b'^2}{a}=\frac{a^2\,(1-e'^2)}{a}.$$

Also $\qquad p'=r\sin(90-\phi)=r\cos\phi.$

$$\therefore \quad v^2r^2\cos^2\phi=\mu a\,(1-e'^2) \qquad \ldots(3)$$

Now we have to find ϕ.

We know that $SY.HZ=b^2$ (original ellipse)

or $\qquad SP\sin\phi\,.\,HP\sin\phi=b^2.$

If $SP=r$, then $HP=2a-r$ as $SP+HP=2a$.

$$\therefore \quad r \sin \phi . (2a-r) \sin \phi = b^2$$

or
$$\sin^2 \phi = \frac{b^2}{r(2a-r)}.$$

$$\therefore \quad \cos^2 \phi = 1 - \sin^2 \phi = \frac{r(2a-r)-b^2}{r(2a-r)} \qquad \ldots(4)$$

Putting for v^2 and $\cos^2 \phi$ from (1) and (4) in (3), we get

$$\mu . \frac{2a-r}{ar} . r^2 . \frac{2ar-r^2-b^2}{r(2a-r)} = \mu a (1-e'^2)$$

or
$$2ar-r^2-b^2=a^2-a^2e'^2$$

or
$$a^2e'^2=(a-r)^2+b^2. \qquad \ldots(5)$$

Now by cosine formula, we get

$$SP^2=CS^2+CP^2+2CS . CP \cos \theta, \qquad [\triangle SPC]$$
$$HP^2=CH^2+CP^2+2CH . CP \cos (180-\theta). \ [\triangle HPC]$$

But $SP=r$; $\therefore$ $HP=2a-r$ and $CS=SH=ae$.

Adding, we get

$$r^2+(2a-r)^2=2a^2e^2+2CP^2,$$
$$2r^2-4ar+4a^2=2a^2e^2+2CP^2$$

or
$$CP^2=a^2-2ar+r^2+a^2-a^2e^2$$

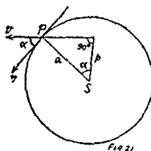
or
$$CP^2=(a-r)^2+b^2.$$

Hence from (5), we get

$$a^2e'^2=CP^2 ; \quad \therefore \quad e'=\frac{1}{a} CP.$$

Above shows that the eccentricity of the new ellipse varies as the distance of P from the centre C.

Ex. 21. *A particle is moving freely in a circle under the inverse square law of attraction towards the centre When it is at a point P of the circle, the direction of motion is turned through an angle α without change of velocity. Show that the new orbit is an ellipse of eccentricity sin α and that P is an end of the minor axis.*



If v be the velocity in a circular orbit, then

$$\frac{v^2}{a}=\frac{\mu}{r^2}=\frac{\mu}{a^2}; \quad \therefore \quad v^2=\frac{\mu}{a}. \quad ...(1)$$

Now the direction of velocity at P is turned through an angle α without change in magnitude So for the new orbit at P the velocity is given by $v^2=\frac{\mu}{a}$.

For an ellipse $V^2 < \frac{2\mu}{r}$ or $< \frac{2\mu}{a}$ at $r=a$. Since $\frac{\mu}{a} < \frac{2\mu}{a}$, so that the particle at P moving with velocity $\sqrt{\frac{\mu}{a}}$ will now describe an ellipse with S at its focus and let its semi-major axis be a'.

For an elliptic orbit $v^2=\mu\left[\frac{2}{r}-\frac{1}{a'}\right]$. When $r=a$, $v^2=\frac{\mu}{a}$ given.

$$\therefore \quad \frac{\mu}{a}=\mu\left[\frac{2}{a}-\frac{1}{a'}\right]; \quad \therefore \quad \frac{1}{a}=\frac{1}{a'}, \quad i.e. \quad a'=a.$$

Now $\quad vp=h=\sqrt{(\mu l)}=\sqrt{\{\mu a\,(1-e^2)\}}, \qquad \because \quad a'=a.$

$$\sqrt{\frac{\mu}{a}}.a\cos\alpha=\sqrt{\{\mu a\,(1-e^2)\}}$$

or $\qquad \cos^2\alpha=1-e^2$ or $e^2=1-\cos^2\alpha=\sin^2\alpha$;

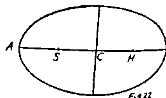
$$\therefore \quad e=\sin\alpha.$$

Hence the particle will describe an ellipse from P with S as the focus and since $SP=a$, therefore P must be the extremity of minor axis as in Q. 8 ; as S is $(ae, 0)$, B is $(0, b)$.

$\therefore \quad SB^2=a^2e^2+b^2=a^2e^2+a^2-a^2e^2=a^2$, *i.e.* $SB=a$, *i.e.* P is at B the extremity of minor axis.

Ex. 22. *A body is describing an ellipse of eccentricity e under the action of a force tending to a focus and when at the nearer apse the centre of force is transferred to the other focus. Prove that the eccentricity of the new orbit is* $\frac{e(3+e)}{1-e}$.

A is the nearer apse with centre of force at S.



$$\therefore \quad r=SA=a(1-e),$$

$$v^2=\mu\left[\frac{2}{r}-\frac{1}{a}\right]=\mu\left[\frac{2}{a(1-e)}-\frac{1}{a}\right]$$

or
$$v^2=\frac{\mu}{a}\left[\frac{1+e}{1-e}\right]. \quad \ldots(1)$$

Now the centre of force is shifted to H the other focus and with this point as centre, r for $A=HA=a(1+e)$. If e' be the eccentricity and $2a'$ the major axis of the new orbit, then velocity at A is given by

$$v^2=\mu\left[\frac{2}{r}-\frac{1}{a'}\right]=\mu\left[\frac{2}{a(1+e)}-\frac{1}{a'}\right]. \quad \ldots(2)$$

Equating the values of v, *i.e.* velocity at A from (1) and (2), we get

$$\mu\left[\frac{2}{a(1+e)}-\frac{1}{a'}\right]=\frac{\mu}{a}\cdot\frac{1+e}{1-e}. \quad \ldots(3)$$

We have seen that for the nearer apse $r=a(1-e)$. Here A is the nearer apse of the new orbit with H as the focus whose eccentricity is e' and semi-major axis a'.

$$\therefore \quad r=a'(1-e')=HA=a(1+e).$$

$$\therefore \quad \frac{1-e'}{a(1+e)}=\frac{1}{a'}.$$

Putting for $\frac{1}{a'}$ in (3), we get

$$\mu\left[\frac{2}{a(1+e)}-\frac{1-e'}{a(1+e)}\right]=\frac{\mu}{a}\frac{1+e}{1-e}$$

or
$$\frac{\mu}{a(1+e)}(1+e')=\frac{\mu}{a}\frac{1+e}{1-e}.$$

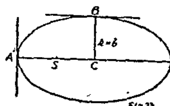
$$\therefore\quad 1+e'=\frac{(1+e)^2}{1-e}\quad\text{or}\quad e'=\frac{(1+e)^2}{1-e}-1$$

or
$$\frac{1+2e+e^2-1+e}{1-e}=\frac{3e+e^2}{1-e}=\frac{e(3+e)}{1-e}.$$

Ex. 23. *A particle is moving in an ellipse of eccentricity e under the acceleration $\frac{\mu}{r^2}$ to the focus. When the particle is nearest to a focus the acceleration is suddenly replaced by an acceleration $\mu' r$ towards the centre of the ellipse. If the particle continues to move in the same ellipse, prove that*

$$\mu=\mu'(1-e)^2 a^3.$$

At A the centre of force is shifted to C and as such A is common to both.



Velocity at A when

$$r=SA=a(1-e)$$

in the elliptic orbit.

$$v^2=\mu\left[\frac{2}{r}-\frac{1}{a}\right]=\mu\left[\frac{2}{a(1-e)}-\frac{1}{a}\right]=\frac{\mu}{a}\cdot\frac{(1+e)}{1-e}. \qquad ...(1)$$

Also at A the direction of motion is perpendicular to SA, *i.e.*, there is an apse at A where $\frac{du}{d\theta}=0$ and the velocity at A is given by (1).

Now the centre of force is C and force is $\mu' r=\frac{\mu'}{u}=P$.

The differential equation of the path is

$$h^2u^2\left[u+\frac{d^2u}{d\theta^2}\right]=P=\frac{\mu'}{u}$$

Ex. 24. (a) *A body describes an ellipse under a force to focus and when at the extremity of minor axis moving towards the nearer apse, it receives a blow in the direction of the other focus which causes it to move towards the centre of the ellipse. Show that the eccentricity of the new orbit is* $(3e^2-3+e^{-2})^{1/2}$ *and that the major axis is turned through an angle whose tangent is* $\dfrac{eb}{a(2e^2-1)}$.

S is the centre of force and $SB=a$.

$\therefore$ velocity at B is given by

$$v^2=\mu\left(\frac{2}{a}-\frac{1}{a}\right)=\frac{\mu}{a}. \quad \ldots(1)$$

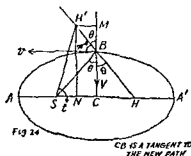


Fig 24

Now at B the particle receives a blow in the direction BH with the result that it begins to move along BC with velocity say V. Now the blow is along BH and hence the velocity **before and after the blow in a direction perpendicular to BH will remain unchanged.** Component of V along $BH=V\cos\theta$ and hence its **component perpendicular to BH is v** $\sin\theta$. Similarly component of v along HB is $v\cos(90-\theta)=v\sin\theta$ and **hence its component perpendicular to HB is v** $\cos\theta$. [Note that focal radii at B are equally inclined to the tangent at B and normal BC at B].

$$\therefore \quad V\sin\theta=v\cos\theta\,; \quad \therefore \quad v=V\tan\theta=V.\frac{CS}{CB}=V.\frac{ae}{b}$$

or
$$\sqrt{\left(\frac{\mu}{a}\right)}=V.\frac{ae}{b} \quad \text{or} \quad \frac{\mu}{a}=V^2.\frac{a^2e^2}{b^2}=\frac{a^2e^2}{a^2(1-e^2)}.$$

$$\therefore \quad V^2=\frac{\mu}{a.a^2e^2}.a^2(1-e^2)=\frac{\mu(1-e^2)}{ae^2}. \quad \ldots(2)$$

Now from B, where $SB=a$, the particle begins to describe new orbit with velocity V and let the major axis of

the new ellipse be $2a'$, so that

$$V^2=\mu\left(\frac{2}{a}-\frac{1}{a'}\right) \quad \text{or} \quad \frac{\mu}{ae^2}(1-e^2)=\mu\left(\frac{2}{a}-\frac{1}{a'}\right) \text{ by (2)}$$

$$\frac{1}{a'}=\frac{1}{a}\left(2-\frac{1-e^2}{e^2}\right)=\frac{3e^2-1}{ae^2}.$$

$$\therefore \quad a'=\frac{ae^2}{3e^2-1}. \qquad \ldots(3)$$

Let e' be the eccentricity of this new ellipse of which BC will be a tangent at B as the particle is moving along BC in this new orbit. If p' be the perpendicular from centre S on the tangent BC to the new ellipse, then $p'=SC=ae$.

Now in any orbit $V.p'=h=\sqrt{(\mu.l')}=\sqrt{[\mu a'(1-e'^2)]}$

or $$\frac{\mu(1-e^2)}{ae^2}.a^2e^2=\mu.\frac{ae^2}{3e^2-1}.(1-e'^2) \text{ by (2) and (3)}$$

or $$\frac{(1-e^2)(3e^2-1)}{e^2}-1=-e'^2$$

or $$e'^2=1-\frac{4e^2-3e^4-1}{e^2}=\frac{3e^4-3e^2+1}{e^2}$$

or $$e'=(3e^2-3+e^{-2})^{1/2}. \qquad \textbf{Proved.}$$

Now we know that in an ellipse tangent at any point is equally inclined to focal radii of that point. Now tangent BC to the new ellipse makes an angle θ with the focal radius SB. Also tangent BC makes an angle θ with HB produced. Hence the other focus will be on HB produced at some point H', so that the focal radii of B are SB and $H'B$ which make equal angles with the tangent CB at B. Hence the line SH' is the line joining the foci of the new ellipse which therefore is the new major axis. Also the original major axis is along SH. Hence if the angle between them be t, then we are to find the value of $\tan t$,

$$\tan t=\frac{H'N}{SN}=\frac{MC}{NS}=\frac{CB+BM}{CS-H'M}=\frac{b+BH'\cos\theta}{ae-BH'\sin\theta}. \qquad \ldots(4)$$

Now $\cos\theta=\frac{CB}{SB}=\frac{b}{a}$ and $\sin\theta=\frac{CS}{SB}=\frac{ae}{a}=e$. ...(5)

Also B is a point on the new ellipse whose foci are S and H', so that $BS+BH'=2a'=$major axis of new ellipse.

$\therefore\quad BH'=2a'-a=\frac{2ae^2}{3e^2-1}-a=\frac{a(1-e^2)}{3e^2-1}$ by (3). ...(6)

Hence from (4) by the help of (5) and (6), we get

$$\tan t=\frac{b+\frac{a(1-e^2)}{3e^2-1}\cdot\frac{b}{a}}{ae-\frac{a(1-e^2)}{3e^2-1}\cdot e}=\frac{b\,.\,2e^2}{ae(4e^2-2)}=\frac{be}{a(2e^2-1)}.$$

Ex. 24. (b) *A particle is describing an ellipse under the action of a force to one of its foci. When the particle is at one extremity of the minor axis, a blow is given to it and subsequent orbit is a circle, find the magnitude and the direction of the blow* **(Punjab 1960)**

Refer figure of part (a).

Before the blow the velocity at B at the end of minor axis is v along the tangent at B *i.e.* $\parallel$ to AA' or $\perp$ to minor axis. Also B is $(0, b)$ and its focal distance is

$$a-ex=a-e\,.\,0=a$$

i.e. $SB=a$. Also in the case of elliptic orbit

$$v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right).$$

$\therefore$ At B where $r=a$, $v^2=\mu\left[\frac{2}{a}-\frac{1}{a}\right]=\frac{\mu}{a}$. ...(1)

After the impulse the path is a circle about S where $SB=a$ will be the radius of the circle and direction of motion will be at right angle to SB. If V the velocity for a circle, then

$$\frac{V^2}{a}=\frac{\mu}{a^2}\quad\text{or}\quad V^2=\frac{\mu}{a}=v^2 \text{ by (1)}$$

$$\therefore\quad V=v.$$

Again suppose (I) is the impulse and its direction makes angles α and β with the directions of v and V. Now the blow has no effect at right angles to its own and hence the momentum at right angles to its direction remain unchanged.

$$\therefore \quad mv \sin \alpha = mV \sin \beta$$

But $v=V$ by (3) $\quad \therefore \quad \alpha=\beta$.

Above shows that the direction of impulse bisects the angle between the directions of v and V. If θ be the angle between SB and BC then the velocities V and v being perpendicular to these will also include an angle θ. Therefore these directions will make an angle $\frac{\theta}{2}$ with the direction of impulse. Resolved parts of v and V in the direction of impulse will be $-v \cos \frac{\theta}{2}$ and $V \cos \frac{\theta}{2}$ (opposite)

$\therefore$ Impulse=change in momentum in its direction

$$= mV \cos \frac{\theta}{2} - \left(-mv \cos \frac{\theta}{2}\right)$$

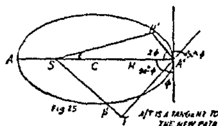
$$= 2mV \cos \frac{\theta}{2} \quad \therefore \quad v=V$$

$$= 2m\sqrt{\frac{\mu}{a}} \sqrt{\left(\frac{1+\cos \theta}{2}\right)} \quad \text{by (2)}$$

But $\qquad \cos \theta = \frac{CB}{SB} = \frac{b}{a}$.

$$\therefore \quad I = m\sqrt{\left[\frac{2\mu}{a}\left(1+\frac{b}{a}\right)\right]} = \frac{m}{a}\sqrt{[2\mu\,(a+b)]}.$$

Ex. 25. *A particle is describing an ellipse of eccentricity e with an acceleration to the focus, when at the further end of major axis its direction of motion is turned through an angle ϕ in the plane of the ellipse, the speed remaining unchanged. Show that the new path is an ellipse of eccentricity $\sqrt{(\sin^2 \phi + e^2 \cos^2 \phi)}$ and that the major axis is turned through an angle whose cosine is* $\dfrac{(\sin^2 \phi + e \cos^2 \phi)}{\sqrt{(\sin^2 \phi + e^2 \cos^2 \phi)}}$.



This question is exactly as Q. 24. S is the centre of force and $SA'=a(1+e)$.

$\therefore$ velocity at A' is given by

$$v^2=\mu\left[\frac{2}{a(1+e)}-\frac{1}{a}\right] \quad ...(1)$$

At A' the direction is changed without change of velocity. If v be the velocity at A' in the new orbit whose semi-major axis is a', then

$$v^2=\mu\left[\frac{2}{a(1+e)}-\frac{1}{a'}\right]. \quad ...(2)$$

From (1) and (2), we find that $a'=a$.

Now $A'T$ is the new direction of tangent at A' which is inclined to original tangent at A' at an angle ϕ. If p' be perpendicular on it from focus, then $p'=ST=SA'\cos\phi$ $=a(1+e)\cos\phi$. Also p the perpendicular on the original tangent from focus S is $SA=a(1+e)$, velocity is unchanged. Applying $vp=h=\sqrt{[\mu a(1-e^2)]}$, we get

$$v.a(1+e)=\sqrt{[\mu a(1-e^2)]} \quad \text{for original,}$$

$$v.a(1+e)\cos\phi=\sqrt{[\mu a'(1-e'^2)]}, \quad \text{for new ellipse.}$$

Dividing, we get $\cos\phi=\sqrt{\left\{\frac{(1-e'^2)}{(1-e^2)}\right\}}$ as $a'=a$

or $$(1-e^2)\cos^2\phi=1-e'^2.$$

$$\therefore \quad e'^2=1-\cos^2\phi+e^2\cos^2\phi=\sin^2\phi+e^2\cos^2\phi.$$

$$\therefore \quad e'=\sqrt{(\sin^2\phi+e^2\cos^2\phi)}.$$

Now in an ellipse tangent is equally inclined to the focal radii. $A'T$ is the tangent to the new path at A' which is inclined to focal radius SA' at an angle $90-\phi$. Hence the other focus will be at H', such that $H'A'$ also makes an angle $90-\phi$ with the tangent TA' produced back. Hence from the figure it is clear that $SA'H'=2\phi$ and the new major axis is SH' inclined to the original major axis at an angle say t.

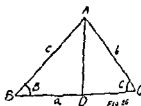
Also sum of the focal distances = major axis.

$$\therefore \quad SA'+H'A'=2a'=2a$$

or $$a(1+e)+H'A'=2a.$$

$$\therefore \quad H'A'=a(1-e).$$

Also SH' = distance between the new foci $=2a'e'=2ae'$.



Now we know that in any triangle ABC,

$$BC=BC+DC=c\cos B+b\cos C$$

or $$a=b\cos C+c\cos B.$$

Applying the above formula in $\triangle SH'A'$, we get

$$SA'=SH'\cos t+H'A'\cos 2\phi,$$

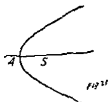
$$a(1+e)=2ae'\cos t+a(1-e)\cos 2\phi.$$

$$\therefore \quad (1+e)-(1-e)\cos 2\phi=2e'\cos t.$$

$$2\sin^2\phi+e.2\cos^2\phi=2\sqrt{(\sin^2\phi+e^2\cos^2\phi)}.\cos t.$$

$$\therefore \quad \cos t=\frac{\sin^2\phi+e\cos^2\phi}{\sqrt{(\sin^2\phi+e^2\cos^2\phi)}}.$$ Hence proved.

Ex. 26. *A comet describing a parabola about the sun, when nearest to it suddenly breaks up, without gain or loss of kinetic energy into two equal portions one of which describes a circle. Prove that the other will describe a hyperbola of eccentricity 2.*



$SA=a$, and in a parabolic orbit the velocity is given by

$$v^2=\frac{2\mu}{r}=\frac{2\mu}{a} \text{ at } A. \qquad ...(1)$$

After explosion let the velocities be v_1 and v_2 at A where $SA=r=a$. One of them describes a circle and we know that velocity for a circle is given by

$$\frac{v_1^2}{a}=\frac{\mu}{r^2}=\frac{\mu}{a^2}; \quad \therefore \quad v_1^2=\frac{\mu}{a}. \qquad ...(2)$$

Also we are given that on account of explosion there is no loss or gain of K. E. and the particle breaks in two equal portions.

$$\therefore \quad \tfrac{1}{2}mv_1^2+\tfrac{1}{2}mv_2^2=\tfrac{1}{2}(m+m)v^2$$

or $$\frac{\mu}{a}+v_2^2=2\cdot\frac{2\mu}{a} \qquad \text{by (1) and (2)}$$

$$\therefore \quad v_2^2=\frac{3\mu}{a}.$$

Now since v_2^2 is $>\frac{2\mu}{r}$ *i.e.* $\frac{2\mu}{a}$, therefore the second part describes a hyperbola. If a' be the semi-transverse axis, then

$$v_2^2=\mu\left[\frac{2}{a}+\frac{1}{a'}\right]$$

or $$\frac{3\mu}{a}=\mu\left[\frac{2}{a}+\frac{1}{a'}\right]; \quad \therefore \quad \frac{1}{a}=\frac{1}{a'} \quad \text{or} \quad a'=a.$$

Also $v_2p=h=\sqrt{(\mu l)}=\sqrt{\{\mu \cdot a'(e^2-1)\}}$ in a hyperbola.

But p at $A=a$ and $a'=a$. Now square and put for v_2^2.

$$\therefore \quad \frac{3\mu}{a}\cdot a^2=\mu\cdot a(e^2-1) \quad \text{or} \quad e^2-1=3 \quad \text{or} \quad e^2=4.$$

$$\therefore \quad e=2. \qquad \text{Hence proved.}$$

Ex. 27. *A particle of mass m when at any point of the ellipse of semi-major axis a is split by an explosion into two particles of masses m_1 and m_2. If the particle m_1 describes a parabola, prove that the semi-major axis of the orbit described by m_2 is $\frac{\mu m_2 a}{\mu m-aE}$ where $\frac{1}{2}E$ is the energy generated by the explosion.*

$$v^2=\mu\left[\frac{2}{r}-\frac{1}{a}\right],$$

$$v_1^2=\mu\cdot\frac{2}{r} \text{ and } v_2^2=\mu\left[\frac{2}{r}-\frac{1}{a'}\right].$$

Also $\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2=\frac{1}{2}mv^2+\frac{1}{2}E$

and $m_1+m_2=m.$

$$\therefore \quad (m-m_2)\frac{2\mu}{r}+m_2.\mu\left[\frac{2}{r}-\frac{1}{a'}\right]=m\mu\left[\frac{2}{r}-\frac{1}{a}\right]+E$$

or $$m_2\mu\left[\frac{2}{r}-\frac{1}{a'}-\frac{2}{r}\right]=m\mu\left[\frac{2}{r}-\frac{1}{a}-\frac{2}{r}\right]+E$$

or $$\frac{m_2\mu}{a'}=\frac{m\mu}{a}-E=\frac{m\mu-aE}{a}.$$

$$\therefore \quad a'=\frac{\mu m_2 a}{m\mu-aE}.$$ Proved.

Ex. 28. *A meteor of mass m describing a parabolic orbit about the sun collides at perihelion with a planet of mass M describing a circular orbit, the bodies moving in the same direction. Prove that if the ratio of m to M is small, the combined bodies will describe an ellipse whose semi-major axis is* $c\left\{1+2(\sqrt{2}-1)\frac{m}{M}\right\}$ *where c is the distance of the bodies from the sun at collision.*

Let v_1 be the velocity of meteor of mass m describing a parabolic orbit and v_2 that of the planet of mass M describing a circular orbit and V be the velocity of the combined body after collision when at a distance c from the sun.

$$\therefore \quad v_1^2=\frac{2\mu}{r},\ v_2^2=\frac{\mu}{r}=\frac{\mu}{c} \quad \therefore \quad v_1^2=2v_2^2. \qquad \ldots(1)$$

Also $(m+M)V=mv_1+Mv_2=v_2(m\sqrt{2}+M)$ by (1).

$$\therefore \quad V=v_2\frac{M+m\sqrt{2}}{m+M}=v_2\left(1+\sqrt{2}\frac{m}{M}\right)\left(1+\frac{m}{M}\right)^{-1}$$

or $$V=v_2\left(1+\sqrt{2}\frac{m}{M}\right)\left(1-\frac{m}{M}+\ldots\right) \text{ as } \frac{m}{M} \text{ is small}$$

or $$V=v_2\left[1+(\sqrt{2}-1)\frac{m}{M}\right]$$

or $$V^2=v_2^2\left[1+(\sqrt{2}-1)\frac{m}{M}\right]^2$$

or $$V^2=\frac{\mu}{c}\left[1+2(\sqrt{2}-1)\frac{m}{M}\right]. \quad ...(1)$$

Clearly the expression within brackets on the R. H. S. is less than 2 and hence $V^2<\frac{2\mu}{c}$ so that the combined body will describe an ellipse. Let a' be the semi-major axis of the orbit.

$$\therefore \quad V^2=\mu\left[\frac{2}{r}-\frac{1}{a'}\right]=\mu\left[\frac{2}{c}-\frac{1}{a'}\right] \text{ when } r=c$$

or $$\frac{\mu}{c}\left[1+2(\sqrt{2}-1)\frac{m}{M}\right]=\mu\left[\frac{2}{c}-\frac{1}{a'}\right]$$

or $$\frac{1}{a'}=\frac{1}{c}\left[1-2(\sqrt{2}-1)\frac{m}{M}\right].$$

$$\therefore \quad a'=c\left[1-2(\sqrt{2}-1)\frac{m}{M}\right]^{-1}=c\left[1+2(\sqrt{2}-1)\frac{m}{M}\right].$$

Ex. 29. *A body is describing an ellipse under a force to the focus When it is at the end of minor axis, an internal explosion occurs which acts in the direction of motion and breaks it up into two equal masses. If one of them describes a parabola and the other an ellipse, show that the ratio of its major axis to that of the original ellipse is* $\sqrt{2}+1:4$

We have done before that for point B, $r=SB=a$.

$$\therefore \quad V^2=\mu\left[\frac{2}{a}-\frac{1}{a}\right]=\frac{\mu}{a}$$

$$V_1^2=\frac{2\mu}{a} \text{ and } V_2^2=\mu\left[\frac{2}{a}-\frac{1}{a'}\right]!$$

Also $$(m+m)\,V=mV_1+mV_2$$

or $$2V=V_1+V_2$$

or $$2\sqrt{\left(\frac{\mu}{a}\right)}=\sqrt{\left(\frac{2\mu}{a}\right)}+\sqrt{\mu}\left[\frac{2}{a}-\frac{1}{a'}\right]^{1/2}$$

Also $\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2=\frac{1}{2}mv^2+\frac{1}{2}E$

and $m_1+m_2=m.$

$$\therefore \quad (m-m_2)\frac{2\mu}{r}+m_2.\mu\left[\frac{2}{r}-\frac{1}{a'}\right]=m\mu\left[\frac{2}{r}-\frac{1}{a}\right]+E$$

or $$m_2\mu\left[\frac{2}{r}-\frac{1}{a'}-\frac{2}{r}\right]=m\mu\left[\frac{2}{r}-\frac{1}{a}-\frac{2}{r}\right]+E$$

or $$\frac{m_2\mu}{a'}=\frac{m\mu}{a}-E=\frac{m\mu-aE}{a}.$$

$$\therefore \quad a'=\frac{\mu m_2\, a}{m\mu-aE}.$$ Proved.

Ex. 28. *A meteor of mass m describing a parabolic orbit about the sun collides at perihelion with a planet of mass M describing a circular orbit, the bodies moving in the same direction. Prove that if the ratio of m to M is small, the combined bodies will describe an ellipse whose semi-major axis is* $c\left\{1+2(\sqrt{2}-1)\frac{m}{M}\right\}$ *where c is the distance of the bodies from the sun at collision.*

Let v_1 be the velocity of meteor of mass m describing a parabolic orbit and v_2 that of the planet of mass M describing a circular orbit and V be the velocity of the combined body after collision when at a distance c from the sun.

$$\therefore \quad v_1^2=\frac{2\mu}{r},\ v_2^2=\frac{\mu}{r}=\frac{\mu}{c} \quad \therefore \quad v_1^2=2v_2^2. \qquad \ldots(1)$$

Also $(m+M)V=mv_1+Mv_2=v_2(m\sqrt{2}+M)$ by (1).

$$\therefore \quad V=v_2\frac{M+m\sqrt{2}}{m+M}=v_2\left(1+\sqrt{2}\frac{m}{M}\right)\left(1+\frac{m}{M}\right)^{-1}$$

or $$V=v_2\left(1+\sqrt{2}\frac{m}{M}\right)\left(1-\frac{m}{M}+\ldots\right) \text{ as } \frac{m}{M} \text{ is small}$$

or $$V=v_2\left[1+(\sqrt{2}-1)\frac{m}{M}\right]$$

or $$V^2 = v_2^2 \left[1+(\sqrt{2}-1)\frac{m}{M}\right]^2$$

or $$V^2 = \frac{\mu}{c}\left[1+2(\sqrt{2}-1)\frac{m}{M}\right]. \quad ...(1)$$

Clearly the expression within brackets on the R. H. S. is less than 2 and hence $V^2 < \frac{2\mu}{c}$ so that the combined body will describe an ellipse. Let a' be the semi-major axis of the orbit.

$$\therefore \quad V^2 = \mu\left[\frac{2}{r}-\frac{1}{a'}\right] = \mu\left[\frac{2}{c}-\frac{1}{a'}\right] \text{ when } r=c$$

or $$\frac{\mu}{c}\left[1+2(\sqrt{2}-1)\frac{m}{M}\right] = \mu\left[\frac{2}{c}-\frac{1}{a'}\right]$$

or $$\frac{1}{a'} = \frac{1}{c}\left[1-2(\sqrt{2}-1)\frac{m}{M}\right].$$

$$\therefore \quad a' = c\left[1-2(\sqrt{2}-1)\frac{m}{M}\right]^{-1} = c\left[1+2(\sqrt{2}-1)\frac{m}{M}\right].$$

Ex. 29. *A body is describing an ellipse under a force to the focus. When it is at the end of minor axis, an internal explosion occurs which acts in the direction of motion and breaks it up into two equal masses. If one of them describes a parabola and the other an ellipse, show that the ratio of its major axis to that of the original ellipse is $\sqrt{2}+1 : 4$*

We have done before that for point B, $r = SB = a$.

$$\therefore \quad V^2 = \mu\left[\frac{2}{a}-\frac{1}{a}\right] = \frac{\mu}{a}$$

$$V_1^2 = \frac{2\mu}{a} \text{ and } V_2^2 = \mu\left[\frac{2}{a}-\frac{1}{a'}\right]!$$

Also $$(m+m)\,V = mV_1 + mV_2$$

or $$2V = V_1 + V_2$$

or $$2\sqrt{\left(\frac{\mu}{a}\right)} = \sqrt{\left(\frac{2\mu}{a}\right)} + \sqrt{\mu}\left[\frac{2}{a}-\frac{1}{a'}\right]^{1/2}$$

or $$\frac{2-\sqrt{2}}{\sqrt{a}}=\left[\frac{2}{a}-\frac{1}{a'}\right]^{1/2}.$$

Squaring,

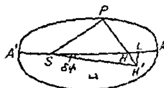
$$\frac{6-4\sqrt{2}}{a}=\frac{2}{a}-\frac{1}{a'} \quad \text{or} \quad \frac{1}{a'}=\frac{2-6+4\sqrt{2}}{a}=\frac{4(\sqrt{2}-1)}{a}.$$

$$\therefore \quad \frac{a'}{a}=\frac{1}{4(\sqrt{2}-1)}=\frac{\sqrt{2}+1}{4(\sqrt{2}-1)(\sqrt{2}+1)}=\frac{\sqrt{2}+1}{4}.$$

§ 5. (b) Effect on account of a small disturbing force along the tangent.

To determine the change in the motion of an elliptic central orbit on account of a sudden impulse along the tangent.

(Delhi Hons. 63, 64)



Let S be the centre of force and P the position of the particle in its orbit when its velocity v along the tangent is changed to $v+\delta v$ along the tangent because of a certain disturbing force along the tangent. Let $2a$ and $2a'$ be the major axis before and after the disturbing the force.

$$\therefore \quad v^2=\mu\left[\frac{2}{r}-\frac{1}{a}\right]=\mu\left[\frac{2}{SP}-\frac{1}{a}\right] \qquad ...(1)$$

$$(v+\delta v)^2=\mu\left[\frac{2}{r}-\frac{1}{a'}\right]=\mu\left[\frac{2}{SP}-\frac{1}{a'}\right]. \qquad ...(2)$$

Differentiating (1), we get

$$2v\,\delta v=\mu\cdot\frac{1}{a^2}\,\delta a. \qquad (SP \text{ is constant})$$

$\therefore$ Increase in semi-major axis *i.e.* δa is given by

$$\delta a=\frac{2v\,\delta v\,a^2}{\mu}. \qquad ...(3)$$

(Delhi Hons. 63, 64)

Again the direction of motion at P remains unchanged and as such the perpendicular from S on the tangent at P *i.e.* p remains unchanged.

$\therefore$ Also $vp=h$, $\therefore$ $p\,\delta v=\delta h$ or $\frac{h}{v}\,\delta v=\delta h$. ...(4)

Change in eccentricity.

$$h^2=\mu l=\mu . a\,(1-e^2).$$

Differentiate $2h\,\delta h=\mu\,[(1-e^2)\,\delta a-2ae\,\delta e]$.

Putting for δh and δa from (3) and (4) in the above, we get

$$2\mu\,ae\,\delta e=\mu\,(1-e^2)\,\frac{2v\,a^2}{\mu}\,\delta v-2h.\frac{h}{v}\,\delta v$$

$$=2v\,a^2\,(1-e^2)\,\delta v-\frac{2\mu\,a\,(1-e^2)}{v}\,\delta v$$

$$=2a\,(1-e^2)\,\delta v\left[av-\frac{\mu}{v}\right].$$

$$\therefore\quad \delta e=\frac{\delta v}{v}\,\frac{(1-e^2)}{e}\,.\frac{av^2-\mu}{\mu} \quad \text{...(5)}$$

(Delhi Hons. 63, 64)

Above gives us the change in eccentricity.

Now the direction of motion at P remains unchanged therefore the other focus H' will lie on PH after disturbance

$$HH'=H'P-HP=(H'P+SP)-(HP+SP)$$

or

$$HH'=2a'-2a=2\delta a. \quad \text{...(6)}$$

Now if $\delta\psi$ be the angle HSH' through which the major axis turns, then

$$\delta\psi=\tan\,\delta\psi=\frac{H'L}{SL}=\frac{HH'\,\sin H}{2ae+HH'\cos H}, \qquad \because\ SH=2ae$$

or $\delta\psi=\frac{2\,\delta a\,.\sin H}{2ae}$ as HH' is small

or $\delta\psi=\frac{2v\,\delta v\,a^2}{\mu}\,.\,\frac{\sin H}{ae}=\frac{2va}{e\mu}\,\sin H\,\delta v.$ ...(7)

Again we know that if T be the period, then

$$T=\frac{2\pi}{\sqrt{\mu}}\,a^{3/2}.$$

Defferentiating $\log T = \log \dfrac{2\pi}{\sqrt{\mu}} + \dfrac{3}{2} \log a.$

Differentiating, we get

$$\frac{1}{T}\,\delta T = 0 + \frac{3}{2}\cdot\frac{1}{a}\,\delta a.$$

$$\therefore\quad \delta T = \frac{3T}{2a}\cdot\frac{2v\,\delta v\,.\,a^2}{\mu} = \frac{3va}{\mu}\,\delta v. \qquad \ldots(8)$$

All the above results give us the necessary changes in the various elements.

Ex. 30. *When a periodic comet is at its greatest distance from the sun, its velocity is increased by a small quantity δv. Show that the comet's least distance from the sun is increased by* $4\delta v \left\{\dfrac{a^3\,(1-e)}{\mu\,(1+e)}\right\}^{1/2}$.

$$v^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right]$$

where r=greatest distance from focus

$= a + ae.$

$$\therefore\quad v^2 = \mu\left[\frac{2}{a\,(1+e)} - \frac{1}{a}\right] = \frac{\mu}{a}\,\frac{1-e}{1+e}. \qquad \ldots(1)$$

Also for § 5 (b) result (3) and (5)

$$\delta a = \frac{2v\,\delta v\,.\,a^2}{\mu}$$

and

$$\delta e = \frac{\delta v}{v}\cdot\frac{1-e^2}{e}\;\frac{av^2-\mu}{\mu}.$$

But $\dfrac{av^2}{\mu} - 1 = \dfrac{1-e}{1+e} - 1 = \dfrac{-2e}{1+e}$ by (1),

$$\therefore\quad \delta e = \frac{-2e}{1+e}\cdot\frac{\delta v}{v}\cdot\frac{1-e^2}{e} = -2\,\frac{\delta v}{v}\,(1-e).$$

Again if r be the least distance from focus the $r = a - ae$.

$\therefore$ Change is least distance r is

$$\delta r = (1-e)\,\delta a - a\,\delta e.$$

Put for δa and δe as found above.

$$\therefore \quad \delta r = (1-e)\frac{2v\,\delta v}{\mu}\;a^2 + a\,.\frac{2\delta v}{v}\,(1-e)$$

$$= 2a\,\delta v\,(1-e)\left[\frac{av}{\mu}+\frac{1}{v}\right]$$

$$= 2a\,\delta v\,(1-e)\,\frac{(av^2+\mu)}{\mu v}.$$

Now put for v from (1).

$$\therefore \quad \delta r = \frac{2a\,\delta v\,(1-e)}{\mu}\sqrt{\left(\frac{a}{\mu}\,\frac{1+e}{1-e}\right)}\left[\frac{\mu\,(1-e)}{1+e}+\mu\right]$$

$$= 2a\,\delta v\,(1-e)\sqrt{\left(\frac{a}{\mu}\,\frac{1+e}{1-e}\right)}\left[\frac{2}{1+e}\right]$$

$$= 4\,\delta v\,.\left[\frac{a^3}{\mu}\;\frac{1-e}{1+e}\right]^{1/2}.$$

§ 6. *An important result of Integral Calculus.*

(i) $\displaystyle\int \frac{dx}{a+b\cos x},\; a > b.$

$$I = \frac{dx}{a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

divide above and below by $\cos^2\frac{x}{2}$.

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{(a+b)+(a-b)\tan^2\frac{x}{2}},\; a > b.\; \because\; a-b = +\text{ive}.$$

Put $\tan\frac{x}{2}$; $\therefore$ $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$.

$$= \frac{1}{(a-b)}\int \frac{2\,dt}{\left[\sqrt{\left(\frac{a+b}{a-b}\right)}\right]^2 + t^2}$$

$$=\frac{2}{a-b}\cdot\frac{1}{\sqrt{\left(\frac{a+b}{a-b}\right)}}\tan^{-1}\frac{t}{\sqrt{\left(\frac{a+b}{a-b}\right)}}$$

$$=\frac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\left[\sqrt{\left(\frac{a-b}{a+b}\right)}.\tan\frac{x}{2}\right].$$

If we put $a=1$, $b=e$, where $e<1$, *i.e.* $a>b$, then

$$\int\frac{dx}{1+e\cos x}=\frac{2}{\sqrt{(1-e^2)}}\left[\tan^{-1}\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{x}{2}\right],\ e<1.$$

(ii) $\int\frac{dx}{a+b\cos x}$, $a<b$.

Proceeding as above, we arrive at

$$I=\int\frac{\sec^2\frac{x}{2}\,dx}{(a+b)-(b-a)\tan^2\frac{x}{2}},\ b>a.\quad\because\ b-a\text{ is +ive.}$$

$$=\frac{1}{(b-a)}\int\frac{2\,dt}{\left[\sqrt{\left(\frac{b+a}{b-a}\right)^2}-t^2\right]},\ \text{where } t=\tan\frac{x}{2}$$

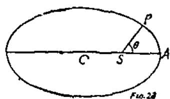
$$=\frac{2}{b-a}\cdot\frac{1}{2\sqrt{\left(\frac{b+a}{b-a}\right)}}\log\frac{\sqrt{\left(\frac{b+a}{b-a}\right)}+t}{\sqrt{\left(\frac{b+a}{b-a}\right)}-t}$$

$$=\frac{1}{\sqrt{(b^2-a^2)}}\log\frac{\sqrt{(b+a)}+\sqrt{(b-a)}\tan\frac{x}{2}}{\sqrt{(b+a)}-\sqrt{(b-a)}\tan\frac{x}{2}}.$$

If we put $a=1$, $b=e$, where $e>1$, *i.e.* $b>a$, then

$$\int\frac{dx}{1+e\cos x}=\frac{1}{\sqrt{(e^2-1)}}\log\frac{\sqrt{(e+1)}+\sqrt{(e-1)}\tan\frac{x}{2}}{\sqrt{(e+1)}-\sqrt{(e-1)}\tan\frac{x}{2}},$$
$$e>1.$$

§ 7. To find the time of description of a given arc of an elliptic orbit starting from the nearer end of the major axis. **(Patna 57, Agra 46)**



Taking the focus S as pole and SA as initial line, the polar equation of the ellipse is

$$\frac{l}{r}=1+e\cos\theta,$$

where $e<1$ and l is the semi-latus rectum.

Also we know that

$$h=r^2\frac{d\theta}{dt}=x\frac{dy}{dt}-y\frac{dx}{dt}\text{ in cartesian,}$$

where x and y are w. r. t. S as origin.

$$\therefore\quad \int_0^t h\,dt=\int r^2\,d\theta=\int_0^\theta\frac{l^2}{(1+e\cos\theta)^2}\,d\theta.\ldots(1)$$

$$\text{Now }\frac{d}{d\theta}\cdot\frac{\sin\theta}{1+e\cos\theta}=\frac{(1+e\cos\theta)\cos\theta-\sin\theta\,(-e\sin\theta)}{(1+e\cos\theta)^2}$$

$$=\frac{\cos\theta+e}{(1+e\cos\theta)^2}=\frac{e\cos\theta+e^2}{e\,(1+e\cos\theta)^2}$$

$$=\frac{(1+e\cos\theta)-(1-e^2)}{e\,(1+e\cos\theta)^2},\ e<1$$

or
$$\frac{d}{d\theta}\frac{\sin\theta}{1+e\cos\theta}=\frac{1}{e\,(1+e\cos\theta)}-\frac{1-e^2}{e}\cdot\frac{1}{(1+e\cos\theta)^2}.$$

Integrating both sides, we get

$$\frac{1-e^2}{e}\int\frac{1}{(1+e\cos\theta)^2}\,d\theta=\frac{1}{e}\int\frac{d\theta}{1+e\cos\theta}-\frac{\sin\theta}{1+e\cos\theta}.$$

Multiplying throughout by $\dfrac{l^2.e}{1-e^2}$, we get

$$\int\frac{l^2}{(1+e\cos\theta)^2}\,d\theta=\frac{l^2}{1-e^2}\left[\frac{2}{\sqrt{(1-e^2)}}\tan^{-1}\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}\right]$$
$$-\frac{l^2.e}{1-e^2}\,\frac{\sin\theta}{1+e\cos\theta}\qquad\ldots(2)$$

(by § 6).

Also $l=\frac{b^2}{a}=\frac{a^2(1-e^2)}{e}=a(1-e^2)$ ...(3)

and $h=\sqrt{(\mu l)}=\sqrt{\{\mu . a(1-e^2)\}}$. ...(4)

Hence from (1), (2), (3) and (4), we get

$$\sqrt{[\mu a(1-e^2)]}\,.\,t=\frac{2a^2(1-e^2)^2}{(1-e^2)^{3/2}}\tan^{-1}\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}$$
$$-\frac{a^2(1-e^2)^2 e}{1-e^2}\,\frac{\sin\theta}{1+e\cos\theta}.$$

Also when $\theta=0$, the R. H. S. of above vanishes.

$$\therefore\quad t=\frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}-e\sqrt{(1-e^2)}\,\frac{\sin\theta}{1+e\cos\theta}\right].$$

§ 8. **To find the time of description of a given arc of a hyperbolic orbit.**

The equation of the hyperbola is

$$\frac{l}{r}=1+e\cos\theta,\ \text{where } e>1.$$

Also $h=r^2\frac{d\theta}{dt}$; $\therefore\ \int_0^t h\,dt=\int_0^\theta \frac{l^2}{(1+e\cos\theta)^2}\,d\theta.$ ...(1)

Also as in § 7,

$$\frac{d}{d\theta}\,\frac{\sin\theta}{1+e\cos\theta}=\frac{\cos\theta+e}{(1+e\cos\theta)^2}=\frac{e\cos\theta+e^2}{e(1+e\cos\theta)^2}$$
$$=\frac{(1+e\cos\theta)+(e^2-1)}{e(1+e\cos\theta)^2},\quad \because\ e>1.$$

$$\therefore\ \frac{d}{d\theta}\,\frac{\sin\theta}{1+e\cos\theta}=\frac{1}{e(1+e\cos\theta)}+\frac{e^2-1}{e}\cdot\frac{1}{(1+e\cos\theta)^2}.$$

Integrating both sides, we get

$$\frac{e^2-1}{e}\int\frac{1}{(1+e\cos\theta)^2}\,d\theta=\frac{\sin\theta}{1+e\cos\theta}-\frac{1}{e}\int\frac{d\theta}{(1+e\cos\theta)}.$$

$$\therefore\ \int\frac{l^2}{(1+e\cos\theta)^2}\,d\theta=\frac{l^2e}{e^2-1}\,\frac{\sin\theta}{1+e\cos\theta}$$
$$-\frac{l^2}{e^2-1}\left[\frac{1}{\sqrt{(e^2-1)}}\log\frac{\sqrt{(e+1)}+\sqrt{(e-1)}\tan\theta/2}{\sqrt{(e+1)}-\sqrt{(e-1)}\tan\theta/2}\right]\quad ...(2)$$

[§ 6]

Now $$l=\frac{b^2}{a}=\frac{a^2(e^2-1)}{e}=a(e^2-1), \qquad \ldots(3)$$

$$h=\sqrt{(\mu l)}=\sqrt{[\mu a(e^2-1)]}. \qquad \ldots(4)$$

Hence from (1), (2), (3) and (4), we get

$$\sqrt{[\mu a(e^2-1)]}\,.\,t=\frac{a^2(e^2-1)^2\; e}{e^2-1}\;\frac{\sin\theta}{1+e\cos\theta}$$

$$-\frac{a^2(e^2-1)^2}{(e^2-1)^{3/2}}\log\left[\frac{\sqrt{(e+1)}+\sqrt{(e-1)}\tan\theta/2}{\sqrt{(e+1)}-\sqrt{(e-1)}\tan\theta/2}\right].$$

$$\therefore\quad t=\frac{a^{3/2}}{\sqrt{\mu}}\left[e\sqrt{(e^2-1)}\;\frac{\sin\theta}{1+e\cos\theta}\right.$$

$$\left.-\log\frac{\sqrt{(e+1)}+\sqrt{(e-1)}\tan\theta/2}{\sqrt{(e+1)}-\sqrt{(e-1)}\tan\theta/2}\right].$$

§ 9. To find the time of description of a given arc of a parabolic orbit starting from the vertex.

(Sagar 64 65 ; Agra 50, 57 ; Punjab 57)

Polar equation of the parabola referred to focus as pole and axis as initial line is

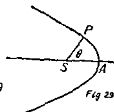


Fig 29

$$\frac{l}{r}=1+\cos\theta,\; e=1,\; h=r^2\frac{d\theta}{dt}.$$

$$\therefore\quad \int_0^t h\,dt=\int_0^\theta r^2\,d\theta=\int_0^\theta \frac{l^2}{(1+\cos\theta)^2}\,d\theta$$

or $$ht=\int_0^\theta \frac{l^2}{4\cos^4\theta/2}\,d\theta=\frac{l^2}{4}\int \sec^2\frac{\theta}{2}\sec^2\frac{\theta}{2}\,d\theta$$

$$=\frac{l^2}{4}\int(1+t^2)\,.\,2\,dt,\text{ where }\tan\frac{\theta}{2}=t$$

or $$ht=\frac{l^2}{2}\left[t+\frac{t^3}{3}\right]=\frac{l^2}{2}\left[\tan\frac{\theta}{2}+\frac{1}{3}\tan^3\frac{\theta}{2}\right].$$

But $h=\sqrt{(\mu l)}$.

$$\therefore\quad t=\frac{l^{3/2}}{2\sqrt{\mu}}\left[\tan\frac{\theta}{2}+\frac{1}{3}\tan^3\frac{\theta}{2}\right],\text{ where } l=2a.$$

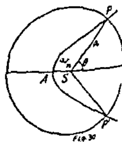
Ex. 1. *The perihelion distance of a planet describing a parabolic orbit is* $\frac{1}{n}$ *of the radius of the earth's path supposed*

circular. Show that the time that the comet will remain within the earth's orbit is

$$\frac{2}{3\pi}\cdot\frac{n+2}{n}\sqrt{\left(\frac{n-1}{2n}\right)} \text{ of a year.}$$

Also prove that the longest time that the comet can remain within the earth's orbit is $\frac{2}{3\pi}$ *of an year.*

(Sagar 64 ; Agra 51 57, 64 ; Punjab 56)



The polar equation of a parabola referred to S as pole is $\frac{l}{r}=1+\cos\theta$, where l is semi-latus rectum.

Since perihelion distance of the planet is $\frac{a}{n}$, where a is the radius of the earth. Hence $SA=\frac{a}{n}$.

Therefore latus rectum of the parabola$=4.SA=\frac{4a}{n}$,

i.e. $\qquad l=\text{semi-latus rectum}=\frac{2a}{n}. \qquad \ldots(1)$

Also $$\cos\theta=\frac{l-r}{r}$$

or $$\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}=\frac{l-r}{r}.$$

Applying componendo and dividendo,

$$\frac{2\tan^2\frac{\theta}{2}}{2}=\frac{2r-l}{l}=\frac{2r-\frac{2a}{n}}{\frac{2a}{n}}=r.\frac{n}{a}-1.$$

Now at the position P, where its path intersects the earth's orbit, we have $\quad r=SP=a$.

$$\therefore \quad \tan^2 \frac{\theta}{2} = (n-1) \quad \text{or} \quad \tan \frac{\theta}{2} = \sqrt{(n-1)}. \qquad \ldots(2)$$

Hence the required time is twice the time of describing an angle θ given by (2).

Now by § 9, $2t = 2 \dfrac{l^{3/2}}{2\sqrt{\mu}} \left(\tan \dfrac{\theta}{2} + \dfrac{1}{3} \tan^3 \dfrac{\theta}{2} \right)$.

Putting for l and $\tan \dfrac{\theta}{2}$ from (1) and (2),

$$\text{required time} = \frac{2 \cdot \left(\frac{2a}{n}\right)^{3/2}}{2\sqrt{\mu}} \left[\sqrt{(n-1)} + \tfrac{1}{3}(n-1)\sqrt{(n-1)}\right]$$

$$= 2 \sqrt{\left(\frac{2a^3}{n^3\mu}\right)} \sqrt{(n-1)} \left(1 + \frac{n-1}{3}\right)$$

$$= \frac{2}{3} \sqrt{\left(\frac{2a^3}{n^3\mu}\right)} (n+2)\sqrt{(n-1)}. \qquad \ldots(3)$$

Now one year $=$ time of revolution of earth $= \dfrac{2\pi}{\sqrt{\mu}} a^{3/2}$.

$$\therefore \quad \sqrt{\frac{a^3}{\mu}} = \frac{\text{one year}}{2\pi}. \quad \text{Put in (3).}$$

$$\therefore \quad \text{Required time} = \frac{2}{3} \sqrt{2} \cdot \frac{\text{one year}}{2\pi} \frac{1}{n\sqrt{n}} (n+2)\sqrt{(n-1)}$$

$$= \frac{2}{3\pi} \frac{(n+2)}{n} \sqrt{\left(\frac{n-1}{2n}\right)} \text{ of an year.}$$

The above period is longest if

$$z = \frac{(n+2)^2 (n-1)}{n^3} \text{ is max.}$$

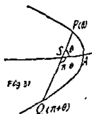
or $\quad z = \dfrac{n^3 + 3n^2 - 4}{n^3} \quad$ or $\quad z = 1 + \dfrac{3}{n} - \dfrac{4}{n^3}$ is longest

for which $\dfrac{dz}{dn} = \dfrac{-3}{n^2} + \dfrac{12}{n^4} = 0 \quad$ or $\quad n^2 = 4 \quad$ or $\quad n = 2$

and $\dfrac{d^2z}{dn^2} = \dfrac{6}{n^3} - \dfrac{48}{n^5} = \dfrac{6}{8} - \dfrac{48}{32} = -$ive for $n = 2$ and hence max.

$\therefore$ Time$=\frac{2}{3\pi}\frac{2+2}{2}\sqrt{\left(\frac{2-1}{2.2}\right)}$ of an year$=\frac{2}{3\pi}$ of an year.

Ex. 2. *Find the time T of describing an arc C of a parabolic orbit under Newtonian Law and if C be bounded by a focal chord, prove that $T \propto$ (focal chord)$^{3/2}$.* (Agra 59)



Proceeding as in § 9 the time of describing an arc starting from vertex is

$$\frac{l^{3/2}}{2\sqrt{\mu}}\left[\tan\frac{\theta}{2}+\frac{1}{3}\tan^3\frac{\theta}{2}\right]$$

where $\frac{l}{r}=1+\cos\theta$ is polar equation of parabola.

If PQ be a focal chord, then if the vectorial angle of P be θ, then that of Q will be $\pi+\theta$ measured anti-clockwise or $-(\pi-\theta)$ measured clockwise.

$$\frac{l}{SP}=1+\cos\theta \quad\text{and}\quad \frac{l}{SQ}=1+\cos(\pi+\theta)=1-\cos\theta.$$

$$\therefore\quad PQ=SP+SQ=\frac{l}{1+\cos\theta}+\frac{l}{1-\cos\theta}=\frac{2l}{\sin^2\theta}$$

$$=2l\operatorname{cosec}^2\theta. \qquad \ldots(1)$$

Therefore the time for QP is the time during which the vectorial angle changes from $-(\pi-\theta)$ to θ, and hence as in § 9,

$$T=\frac{l^{3/2}}{2\sqrt{\mu}}\left[\tan\frac{\theta}{2}+\frac{1}{3}\tan^3\frac{\theta}{2}\right]_{-(\pi-\theta)}^{\theta}$$

$$=\frac{l^{3/2}}{2\sqrt{\mu}}\left[\left(\tan\frac{\theta}{2}+\frac{1}{3}\tan^3\frac{\theta}{2}\right)+\left(\cot\frac{\theta}{2}+\frac{1}{3}\cot^3\frac{\theta}{2}\right)\right]$$

$$=\frac{l^{3/2}}{2\sqrt{\mu}}\left[\frac{\sin^2\theta/2+\cos^2\theta/2}{\sin\theta/2\cos\theta/2}+\frac{1}{3}\frac{\sin^6\theta/2+\cos^6\theta/2}{\sin^3\theta/2\cos^3\theta/2}\right]$$

$$=\frac{l^{3/2}}{2\sqrt{\mu}}\left[\frac{2}{\sin\theta}+\frac{8}{3}\frac{(\sin^2\theta/2+\cos^2\theta/2)(\sin^4\theta/2+\cos^4\theta/2-\cos^2\theta/2\sin^2\theta/2)}{\sin^3\theta}\right]$$

$$=\frac{l^{3/2}}{\sqrt{\mu}}\left[\frac{1}{\sin\theta}+\frac{4}{3}\frac{(1-3\cos^2\theta/2\sin^2\theta/2)}{\sin^3\theta}\right]$$

$$=\frac{l^{3/2}}{\sqrt{\mu}}\left[\frac{3\sin^2\theta+4\ (1-\frac{3}{4}\sin^2\theta)}{3\sin^3\theta}\right]$$

$$=\frac{l^{3/2}}{\sqrt{\mu}}\cdot\frac{4}{3}\operatorname{cosec}^3\theta$$

$=K\,(2l\operatorname{cosec}^2\theta)^{3/2}$ where K is some constant

$=K$ (focal chord)$^{3/2}$ by (1).

Hence T varies as (focal chord)$^{3/2}$.

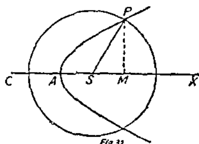
Ex. 3. *A comet moving in the plane of the earth's orbit (supposed circular) describes about the sun a hyperbolic orbit of eccentricity e and its least distance from the sun is 1/n of the radius of earth's orbit. Prove that the time the comet remains within the earth's orbit is* $\frac{T}{\pi}$ $(e\sinh\phi-\phi)$ *where* $e\cosh\phi=1-n+ne$ *and T is the periodic time of the planet describing an elliptic orbit the major axis of which is equal to the transverse axis of the hyperbola.*

Let the equation of the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ so that the co-ordinates of any point on it with C as origin and CX as x-axis are

$(a\cosh\phi,\ b\sinh\phi)$.

Also $CA=a$ and $CS=ae$.

$\therefore\ AS=ae-a$,



The co-ordinates of P referred to S as origin are

$(SM,\ PM)$ i.e. $(CM-CS,\ PM)$

i.e. $x=a\cosh\phi-ae,\ y=b\sinh\phi$.

Now we know that $h=r^2\frac{d\theta}{dt}=x\frac{dy}{dt}-y\frac{dx}{dt}$ where x, y are the co-ordinates of P referred to S as origin.

$\therefore \quad h=(a \cosh \phi - ae)\, b \cosh \phi \dfrac{d\phi}{dt} - b \sinh \phi \,.\, a \sinh \phi \dfrac{d\phi}{dt}.$

$\therefore \quad \int h\, dt = \int [ab (\cosh^2 \phi - \sinh^2 \phi) - abe \cosh \phi]\, d\phi$

or $\qquad ht = ab\,(\phi - e \sinh \phi). \qquad \ldots(1)$

Now $\qquad h^2 = \mu l = \mu \,.\, \dfrac{b^2}{a}.$

$\therefore \quad \sqrt{\mu}\, \dfrac{b}{\sqrt{a}}\, t = ab\,(\phi - e \sinh \phi)$

or $\qquad t = \sqrt{\left(\dfrac{a^3}{\mu}\right)}\,(\phi - e \sinh \phi).$

Now T is the periodic time for an elliptic orbit of major axis $2a$ which is equal to transverse axis.

$$\therefore \quad T = 2\pi \sqrt{\left(\frac{a^3}{\mu}\right)}.$$

$\therefore \quad t = \dfrac{T}{2\pi}(\phi - e \sinh \phi) \quad$ or $\quad 2t = \dfrac{T}{\pi}(\phi - e \sinh \phi).$

Again in a hyperbola focal distance $SP = ex - a$

or $\qquad SP = ea \cosh \phi - a \quad$ or $\quad a_0 = a\,(e \cosh \phi - 1) \quad \ldots(3)$

where a_0 is the radius of the circular orbit.

Also it is given that the least distance from the sun *i.e.* $AS = \dfrac{1}{n}$ of the radius a_0.

$\therefore \quad \dfrac{a_0}{n} = AS = CS - CA = ae - a = a\,(e-1).. \qquad \ldots(4)$

Eliminating a_0 from (3) and (4), we get

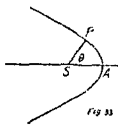
$$a\,(e \cosh \phi - 1) = na\,(e-1)$$

or $\qquad e \cosh \phi = ne - n + 1.$

Ex. 4. *Prove that in a parabolic orbit the time taken to move from the vertex to a point distant r from the focus is* $\dfrac{1}{3\sqrt{\mu}}(r+e)\sqrt{(2r-l)}$ *where 2l is the latus rectum.*

Polar equation of parabola is

$$\frac{l}{r}=1+\cos\theta$$



and as discussed in § 9,

$$t=\frac{l^{3/2}}{2\sqrt{\mu}}\left(\tan\frac{\theta}{2}+\frac{1}{3}\tan^3\frac{\theta}{2}\right)$$

$$=\frac{l^{3/2}}{2\sqrt{\mu}}\tan\frac{\theta}{2}\frac{\left(3+\tan^2\frac{\theta}{2}\right)}{3}. \qquad \ldots(1)$$

Now $\frac{l}{r}=2\cos^2\frac{\theta}{2}$, $\therefore$ $\sec^2\frac{\theta}{2}=\frac{2r}{l}$

or $$\tan^2\frac{\theta}{2}=\sec^2\frac{\theta}{2}-1=\frac{2r-l}{l}. \qquad \ldots(2)$$

$$\therefore \quad t=\frac{l^{3/2}}{2\sqrt{\mu}}\cdot\sqrt{\left(\frac{2r-l}{l}\right)}\left[\frac{3l+2r-l}{3l}\right] \text{ by (1) and (2)}$$

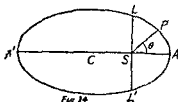
$$=\frac{1}{3\sqrt{\mu}}(r+l)\sqrt{(2r-l)}. \qquad \text{Hence proved.}$$

Ex. 5. *Prove that the time taken to describe the two portions into which an ellipse is divided by the latus rectum through the centre of the force are in the ratio*

$$[\cos^{-1}e-e\sqrt{(1-e^2)}]:[\pi-\cos^{-1}e+e\sqrt{(1-e^2)}].$$

We know from § 7 that the time of describing an arc of an elliptic orbit is given by

$$t=\frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}\right.$$
$$\left.-e\sqrt{(1-e^2)}\frac{\sin\theta}{1+e\cos\theta}\right].$$



Putting $\theta=90°$ and also if $e=\cos z$, then

$$\sqrt{\left(\frac{1-e}{1+e}\right)}=\sqrt{\left(\frac{1-\cos z}{1+\cos z}\right)}=\tan\frac{z}{2}$$

$$\therefore \quad 2\tan^{-1}\sqrt{\left(\frac{1-e}{1+e}\right)}=2\tan^{-1}\tan\frac{z}{2}=2.\frac{z}{2}=z=\cos^{-1}e$$

and $\tan\frac{\theta}{2}=\tan\frac{\pi}{4}=1$ and $\sin 90^\circ=1$, $\cos 90^\circ=0$.

We get the time for describing arc AL as

or
$$t=\frac{a^{3/2}}{\sqrt{\mu}}\left[\cos^{-1}e-e\sqrt{(1-e^2)}\right].$$

$\therefore$ time of describing arc $L'AL$ is $2t$

$$=\frac{2a^{3/2}}{\sqrt{\mu}}\left[\cos^{-1}e-e\sqrt{(1-e^2)}\right]. \quad \ldots(1)$$

Again the total time for describing an elliptic orbit is

$$\frac{2\pi}{\sqrt{\mu}}a^{3/2}.$$

Hence the time of describing the other portion $L'A'L$ is

$$T-2t=\frac{2a^{3/2}}{\sqrt{\mu}}\left[\pi-\cos^{-1}e+e\sqrt{(1-e^2)}\right]. \quad \ldots(2)$$

Hence from (1) and (2), we get the required ratio as given.

Ex. 6. *If the period of a planet be 365 days and the eccentricity* $e=\frac{1}{60}$, *show that the times of describing the two halves of the orbit bounded by the latus rectum through the centre of force are* $\frac{365}{2}\left[1\pm\frac{1}{15\pi}\right]$ *nearly* **(Pb. 57; Agra 54)**

From Q. 5 the time T_1 of describing arc $L'AL$ is

$$T_1=2\frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\sqrt{\left(\frac{1-e}{1+e}\right)}-e\sqrt{(1-e^2)}\right].$$

Since e is small$=\frac{1}{60}$, we neglect e^2 and higher powers.

Also periodic time $T=\frac{2\pi}{\sqrt{\mu}}a^{3/2}=365$ days.

$$\therefore \quad T_1=\frac{T}{\pi}\left[2\tan^{-1}(1-e)^{1/2}(1+e)^{-1/2}-e(1-e^2)^{1/2}\right]$$

$$=\frac{T}{\pi}\left[2\tan^{-1}\left(1-\frac{1}{2}e\ldots\right)\left(1-\frac{1}{2}e\ldots\right)-e\,1-\ldots)\right]$$

$$=\frac{T}{\pi}[2\tan^{-1}(1-e)-e]. \qquad \ldots(1)$$

Now $\tan^{-1}1=\frac{\pi}{4}$; $\therefore \quad \tan^{-1}(1-e)=\frac{\pi}{4}-z$

where z is small.

$$\therefore \quad 1-e=\tan\left(\frac{\pi}{4}-z\right)=\frac{1-\tan z}{1+\tan z}=(1-\tan z)(1+\tan z)^{-1}$$

or $\quad 1-e=(1-\tan z)(1-\tan z\ldots)=(1-2\tan z\ldots)$.

$$\therefore \quad \tan z=\frac{e}{2}=\frac{1}{120}; \quad \therefore \quad z=\tan^{-1}\frac{1}{120}=\frac{1}{120}\text{ approx.}$$

$$\therefore \quad T_1=\frac{T}{\pi}\left[2\left(\frac{\pi}{4}-z\right)-\frac{1}{60}\right]$$

$$=\frac{T}{\pi}\left[\frac{\pi}{2}-\frac{1}{60}-\frac{1}{60}\right]=\frac{T}{2}\left[1-\frac{1}{15\pi}\right]\text{ approximately.}$$

If T_2 be the time of describing $L'A'L$, then

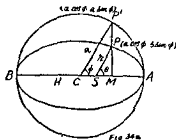
$$T_2=T-T_1=T-\frac{T}{2}\left(1-\frac{1}{15\pi}\right)=\frac{T}{2}\left(1+\frac{1}{15\pi}\right).$$

Hence T_1 and T_2 are given by

$$\frac{T}{2}\left(1\pm\frac{1}{15\pi}\right)=\frac{365}{2}\left(1\pm\frac{1}{15\pi}\right).$$

ANOMALIES.

§ 10. Definitions. (Agra 48, 63)



The planet P describes an ellipse with sun S as the focus whose major axis is $2a$.

Let (r, θ) be the polar co-ordinates of P referred to S as pole. If the cartesian co-ordinates of P referred to C as origin be $(a\cos\phi,\ b\sin\phi)$,

then the co-ordinates of P referred to S as origin will be

$$x = SM = CM - CS = a \cos \phi - ae,$$
$$y = PM = b \sin \phi.$$

Perihelion. The point A on the path of the planet when it is nearest the sun is called *perihelion.*

Aphelion. The point on the path of the planet when it is farthest from the sun is called aphelion.

Eccentric anomaly. The eccentric angle ϕ of any position P of the planet on the ellipse is called the eccentric anomaly. *It is the angle subtended at the centre* by the ordinate of the auxiliary circle through P.

True anomaly. The vectorial angle θ referred to S as pole of any position P of the planet on the ellipse is called true anomaly

Mean angular velocity. If T be periodic time$=\dfrac{2\pi}{\sqrt{(\mu)}}a^{3/2}$ of the planet, then $\dfrac{2\pi}{T}$ denoted by n is called the mean angular velocity. $\therefore$ $n=\dfrac{\sqrt{\mu}}{a^{3/2}}$ or $n^2a^3=\mu=$constant.

Mean anomaly. At any time t, nt is called mean anomaly and is denoted by m. This is equal to the angle which the planet would describe starting from perihelion position if the angular velocity had been constant and equal to mean angular velocity n, *i e.* $\dfrac{2\pi}{T}$.

Heliocentric distance of planet. The radius vector r of the position P of the planet on its path ,is called heliocentric distance from the planet, *i.e* its distance from the sun.

§ **11. Various relations.**

(a) Mean anomaly m in terms of eccentric anomaly ϕ.
(Punjab 56)

Let P be the position of planet after time t, starting from perihelion position A. If its co-ordinates referred to S as origin be x and y, then $x=a\cos\phi-ae$, $y=b\sin\phi$.

Now $h=$twice the rate of description of area.

$$\therefore \quad h=r^2\frac{d\theta}{dt}=x\frac{dy}{dt}-y\frac{dx}{dt}$$

or $h=[(a\cos\phi-ae)\,b\cos\phi-b\sin\phi\,(-a\sin\phi)]\dfrac{d\phi}{dt}$

$$=ab\,[\cos^2\phi+\sin^2\phi-e\cos\phi]\frac{d\phi}{dt}=ab\,[1-e\cos\phi]\frac{d\phi}{dt}.$$

Integrating, we get

$$ht=ab\,(\phi-e\sin\phi)+C.$$

Since when $t=0$, $\phi=0$ at A; $\therefore\ C=0$.

$$\therefore \quad ht=ab(\phi-e\sin\phi). \qquad \ldots(1)$$

If T be the periodic time, then putting $t=T$ and $\phi=2\pi$, we get $\quad hT=ab.\,2\pi\,;\ \therefore\ h=ab.\dfrac{2\pi}{T}=ab.n, \qquad \ldots(2)$

where n is the mean angular velocity.

Putting for h in (1), we get

$$ab.nt=ab\,(\phi-e\sin\phi)$$

or $\qquad nt=\phi-e\sin\phi$. But $m=nt$.

$\therefore\ \ \mathbf{m=\phi-e\sin\phi}$ **or** $\mathbf{\phi=m+e\sin\phi}$. **(Sagar 63)**

Above relation between m and ϕ is called **Kepler's equation.** Here we get m in terms of ϕ.

(b) Eccentric anomaly ϕ in terms of mean anomaly m.

We have proved in part (a) that $m=\phi-e\sin\phi$.

$$\therefore \quad \phi=m+e\sin\phi. \qquad \ldots(1)$$

If e be small, then to a first approximation, $\phi=m$.

In order to find a better approximation, we put $\phi=m$ in (1) and we get $\qquad \phi=m+e\sin m. \qquad \ldots(2)$

then the co-ordinates of P referred to S as origin will be

$$x = SM = CM - CS = a\cos\phi - ae,$$
$$y = PM = b\sin\phi.$$

Perihelion. The point A on the path of the planet when it is nearest the sun is called *perihelion.*

Aphelion. The point on the path of the planet when it is farthest from the sun is called aphelion.

Eccentric anomaly. The eccentric angle ϕ of any position P of the planet on the ellipse is called the eccentric anomaly. *It is the angle subtended at the centre by the ordinate of the auxiliary circle through P.*

True anomaly. The vectorial angle θ referred to S as pole of any position P of the planet on the ellipse is called true anomaly

Mean angular velocity. If T be periodic time$=\frac{2\pi}{\sqrt{(\mu)}}a^{3/2}$ of the planet, then $\frac{2\pi}{T}$ denoted by n is called the mean angular velocity. $\therefore\ n=\frac{\sqrt{\mu}}{a^{3/2}}$ or $n^2a^3=\mu=$constant.

Mean anomaly. At any time t, nt is called mean anomaly and is denoted by m. This is equal to the angle which the planet would describe starting from perihelion position if the angular velocity had been constant and equal to mean angular velocity n, *i e.* $\frac{2\pi}{T}$.

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(a) Mean anomaly m in terms of eccentric anomaly ϕ.
(Punjab 56)

Let P be the position of planet after time t, starting from perihelion position A. If its co-ordinates referred to S as origin be x and y, then $x=a\cos\phi-ae$, $y=b\sin\phi$.

Now $h=$ twice the rate of description of area.

$$\therefore \quad h=r^2\frac{d\theta}{dt}=x\frac{dy}{dt}-y\frac{dx}{dt}$$

or $h=[(a\cos\phi-ae)\,b\cos\phi-b\sin\phi\,(-a\sin\phi)]\,\dfrac{d\phi}{dt}$

$$=ab\,[\cos^2\phi+\sin^2\phi-e\cos\phi]\frac{d\phi}{dt}=ab\,[1-e\cos\phi]\,\frac{d\phi}{dt}.$$

Integrating, we get

$$ht=ab\,(\phi-e\sin\phi)+C.$$

Since when $t=0$, $\phi=0$ at A; $\because\ C=0$

$$\therefore \quad ht=ab(\phi-e\sin\phi). \qquad \ldots(1)$$

If T be the periodic time, then putting $t=T$ and $\phi=2\pi$,

we get $\qquad hT=ab.\,2\pi\,;\ \therefore\ h=ab.\dfrac{2\pi}{T}=ab.n, \qquad \ldots(2)$

where n is the mean angular velocity.

Putting for h in (1), we get

$$ab.nt=ab\,(\phi-e\sin\phi)$$

or $\qquad nt=\phi-e\sin\phi.$ But $m=nt$.

$$\therefore \quad \mathbf{m=\phi-e\sin\phi \quad or \quad \phi=m+e\sin\phi.} \quad \textbf{(Sagar 63)}$$

Above relation between m and ϕ is called Kepler's equation. Here we get m in terms of ϕ.

(b) Eccentric anomaly ϕ in terms of mean anomaly m.

We have proved in part (a) that $m=\phi-e\sin\phi$.

$$\therefore \quad \phi=\mathbf{m+e\sin}\,\phi. \qquad \ldots(1)$$

If e be small, then to a first approximation, $\phi=m$.

In order to find a better approximation, we put $\phi=m$ in (1) and we get $\qquad \phi=m+e\sin m. \qquad \ldots(2)$

Again putting the above value of ϕ in (1), we get

$$\phi=m+e \sin (m+e \sin m)$$

or $$\phi=m+e [\sin m \cos (e \sin m)+\cos m . \sin (e \sin m)].$$

Expand $\cos (e \sin m)$ and $\sin (e \sin m)$ but retain terms only upto e^2 as e is small. $\therefore \cos (e \sin m)=1-\frac{e^2 \sin^2 m}{2!}=1$, because the term of e^2 when multiplied with e will be of e^3,

$$\sin (e \sin m)=e \sin m.$$

$$\therefore \quad \phi=m+e [\sin m . 1+\cos m . (e \sin m)]$$

or $$\phi=m+e \sin m+e^2 \sin m \cos m$$

or $$\phi=m+e \sin m+\tfrac{1}{2}e^2 \sin 2m. \quad \ldots(3)$$

Again if we want a still better approximation, we put the value of ϕ from (3) in (1).

$$\therefore \quad \phi=m+e \sin [m+e \sin m+\tfrac{1}{2}e^2 \sin 2m]$$

$$=m+e [\sin m . \cos (e \sin m+\tfrac{1}{2}e^2 \sin 2m)$$
$$+\cos m \sin (e \sin m+\tfrac{1}{2}e^2 \sin 2m)]$$

Expand as before and retain only upto e^3

$$\therefore \quad \phi=m+e \left[\sin m \left(1-\frac{e^2 \sin^2 m}{2!} \ldots\right)\right.$$
$$\left.+\cos m (e \sin m+\tfrac{1}{2}e^2 \sin 2m) \ldots\right]$$

or $$\phi=m+e \left[\sin m-\frac{e^2}{2} \sin^3 m+e \sin m \cos m\right.$$
$$\left.+\tfrac{1}{2}e^2 \sin 2m \cos m\right].$$

Now $$\sin^3 m=\frac{3 \sin m-\sin 3m}{4}$$

and $$\sin 2m \cos m=\tfrac{1}{2} (\sin 3m+\sin m).$$

$$\therefore \quad \phi=m+e \sin m-\frac{e^3}{8} (3 \sin m-\sin 3m)$$
$$+\frac{e^2}{2} \sin 2m+\frac{e^3}{4} (\sin 3m+\sin m)$$

or $\phi=m+\left(e-\frac{3}{8}e^3+\frac{e^3}{4}\right)\sin m+\frac{e^2}{2}\sin 2m+\left(\frac{e^3}{4}+\frac{e^3}{8}\right)\sin 3m$

or $\phi=m+\left(e-\frac{e^3}{8}\right)\sin m+\frac{e^2}{2}\sin 2m+\frac{3}{8}e^3\sin 3m.$

(c) To express true anomaly θ in terms of eccentric anomaly ϕ.

If (r, θ) and (x, y) be the co-ordinates of P referred to S as pole and origin, then $x=SM=CM-CS=a\cos\phi-ae$.

But $x=r\cos\theta$. $\therefore$ $a\cos\phi-ae=r\cos\theta$.

$$\therefore\quad \cos\theta=\frac{a(\cos\phi-e)}{r}.\qquad\ldots(1)$$

Also $y=r\sin\theta$ or $b\sin\phi=r\sin\theta$.

$\therefore$ $(a\cos\phi-ae)^2+b^2\sin^2\phi=r^2(\cos^2\theta+\sin^2\theta)$

or $a^2\cos^2\phi+a^2e^2-2a^2e\cos\phi+(a^2-a^2e^2)\sin^2\phi=r^2$

or $a^2[(\cos^2\phi+\sin^2\phi)-2e\cos\phi+a^2e^2(1-\sin^2\phi)]=r^2$

or $a^2[1-2e\cos\phi+e^2\cos^2\phi]=r^2.$

$$\therefore\quad r=a(1-e\cos\phi).$$

r=focal distance of P on the ellipse is $a-ex$ where (x, y) are the co-ordinates of P referred to C as origin and is $a\cos\phi$; $\therefore$ $r=a-ae\cos\phi=a(1-e\cos\phi)$.

Putting for r in (1), we get

$$\cos\theta=\frac{a(\cos\phi-e)}{a(1-e\cos\phi)}=\frac{\cos\phi-e}{1-e\cos\phi}.$$

Applying componendo and dividendo, we get

$$\frac{1-\cos\theta}{1+\cos\theta}=\frac{(1-e\cos\phi)-(\cos\phi-e)}{(1-e\cos\phi)+(\cos\phi-e)}=\frac{(1+e)(1-\cos\phi)}{(1-e)(1+\cos\phi)}$$

or
$$\tan^2\frac{\theta}{2}=\frac{1+e}{1-e}\tan^2\frac{\phi}{2}.$$

$$\therefore\quad \tan\frac{\theta}{2}=\sqrt{\left(\frac{1+e}{1-e}\right)}\tan\frac{\phi}{2}.\qquad\ldots(2)$$

or
$$\frac{\sin\frac{\theta}{2}\cos\frac{\phi}{2}}{\cos\frac{\theta}{2}\sin\frac{\phi}{2}}=\sqrt{\left(\frac{1+e}{1-e}\right)}.$$

Applying componendo and dividendo again, we get
$$\frac{\sin\frac{\theta-\phi}{2}}{\sin\frac{\theta+\phi}{2}}=\frac{\sqrt{(1+e)}-\sqrt{(1-e)}}{\sqrt{(1+e)}+\sqrt{(1-e)}}$$

or
$$\frac{\sin\frac{\theta-\phi}{2}}{\sin\frac{\theta+\phi}{2}}=\frac{\{\sqrt{(1+e)}-\sqrt{(1-e)}\}^2}{(1+e)-(1-e)}=\frac{1-\sqrt{(1-e^2)}}{e}$$

$$\therefore\quad \sin\frac{\theta-\phi}{2}=\frac{1-\sqrt{(1-e^2)}}{e}\sin\left(\frac{\theta-\phi}{2}+\phi\right).$$

Let us put $\frac{\theta-\phi}{2}=x$, $\phi=\alpha$, $\frac{1-\sqrt{(1-e^2)}}{e}=n$, so that
$$\sin x=n\sin(x+\alpha).$$

From trigonometry, by expansion, we have
$$x=n\sin\alpha+\frac{n^2}{2}\sin 2\alpha+\frac{n^3}{3}\sin 3\alpha+\ldots \quad \ldots(3)$$

Now $$n=\frac{1-(1-e^2)^{1/2}}{e}=\frac{1-\left[1-\frac{1}{2}e^2+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}e^4\ldots\right]}{e}$$

or
$$n=\tfrac{1}{2}e+\tfrac{1}{8}e^3 \text{ upto } e^3.$$
$$\therefore\quad n^2=\tfrac{1}{4}e^2 \text{ upto } e^3,$$
$$n^3=\tfrac{1}{8}e^3\ldots$$

Putting for x, α, n, n^2, n^3 etc. in (3), we get
$$\frac{\theta-\phi}{2}=(\tfrac{1}{2}e+\tfrac{1}{8}e^3)\sin\phi+\tfrac{1}{2}(\tfrac{1}{4}e^2)\sin 2\phi+\tfrac{1}{3}(\tfrac{1}{8}e^3)\sin 3\phi+\ldots$$
$$\therefore\quad \theta=\phi+(e+\tfrac{1}{4}e^3)\sin\phi+\tfrac{1}{4}e^2\sin 2\phi+\tfrac{1}{12}e^3\sin 3\phi.$$

Also we know from relation (a) result (2) that $h=abn$.

$$\therefore \quad \int_0^t ab\,.n\, dt=\int_0^\theta r^2\, d\theta$$

or
$$abnt=\int_0^\theta \frac{l^2}{(1+e\cos\theta)^2}\, d\theta.$$

$$\therefore \quad nt=m=\frac{l^2}{ab}\int_0^\theta (1+e\cos\theta)^{-2}\, d\theta$$

or
$$m=\frac{b^4}{a^2.ab}\int_0^\theta (1-2e\cos\theta+3e^2\cos^2\theta-4e^3\cos^3\theta\ldots)\, d\theta$$

$$\because \quad l=\frac{b^2}{a}$$

or
$$m=(1-e^2)^{3/2}\int_0^\theta \left[1-2e\cos\theta+3e^2\frac{(1+\cos 2\theta)}{2}\right.$$
$$\left.-e^3(\cos 3\theta+3\cos\theta)\right] d\theta.$$

We have retained terms only upto e^3. Integrating and putting the limits, we get

$$m=\left(1-\frac{3}{2}e^2\ldots\right)\left[\left(1+\frac{3e^2}{2}\right)\theta-(2e+3e^3)\sin\theta\right.$$
$$\left.+\frac{3e^2}{4}\sin 2\theta-\frac{e^3}{3}\sin 3\theta\right].$$

or
$$m=1.\theta-(2e-3e^3+3e^3)\sin\theta+\frac{3e^2}{4}\sin 2\theta-\frac{e^3}{3}\sin 3\theta.$$

$$\therefore \quad \theta=m+2e\sin\theta-3\frac{e^2}{4}\sin 2\theta+\frac{e^3}{3}\sin 3\theta. \qquad \ldots(1)$$

Now we shall eliminate θ from the R. H. S.

To a first approximation, $\theta=m$.

To a second approximation upto e, we put $\theta=m$ and we get
$$\theta=m+2e\sin m.$$

In order to find a better approximation upto e^2, we put $\theta=m+2e\sin m$.

$\therefore \quad \theta = m + 2e \sin(m + 2e \sin m) - \frac{3}{4}e^2 \sin(2m + 4e \sin m).$

Since the last term in R. H. S. is multiplied by e^2, we write it as $\frac{3}{4}e^2 \sin 2m$ only.

$$\therefore \quad \theta = m + 2e\,[\sin m \,.\, \cos(2e \sin m) + \cos m \sin(2e \sin m)] - \tfrac{3}{4}e^2 \sin 2m$$

or $\theta = m + 2e\,[\sin m \,.\, 1 + \cos m \,.\, (2e \sin m)] - \frac{3}{4}e^2 \sin 2m$

or $\theta = m + 2e \sin m + 2e^2 \sin 2m - \frac{3}{4}e^2 \sin 2m$

or $\theta = m + 2e \sin m + \frac{5}{4}e^2 \sin 2m.$ **(Punjab 60; Agra 48, 63)**

To a still better approximation correct upto e^3; we put the above value of θ in the R. H. S. of (1) and remember that we have to retain only upto e^3.

$$\therefore \quad \theta = m + 2e \sin\,[m + 2e \sin m + \tfrac{5}{4}e^2 \sin 2m]$$
$$-\frac{3}{4}e^2 \sin\,[2m + 4e \sin m \text{ only}] + \frac{e^3}{3} \sin\,(3m \text{ only})$$

or
$$\theta = m + 2e\,[\sin m \cos(2e \sin m + \tfrac{5}{4}e^2 \sin 2m) + \cos m \sin(2e \sin m + \tfrac{5}{4}e^2 \sin 2m)]$$
$$-\tfrac{3}{4}e^2\,[\sin 2m \cos(4e \sin m) + \cos 2m \sin(4e \sin m)] + \frac{e^3}{3} \sin 3m$$

or
$$\theta = m + 2e\left[\sin m\left(1 - \frac{4e^2 \sin^2 m}{2}\right)\right] + \cos m\,[2e \sin m + \tfrac{5}{4}e^2 \sin 2m]$$
$$-\frac{3}{4}e^2\,[\sin 2m \,.\, 1 + \cos 2m\,(4e \sin m)] + \frac{e^3}{3} \sin 3m$$

or
$$\theta = m + 2e \sin m - 4e^3 \sin^3 m + 2e^2 \sin 2m + \tfrac{5}{2}e^3 \sin 2m \cos m$$
$$-\frac{3}{4}e^2 \sin 2m - 3e^3 \sin m \cos 2m + \frac{e^3}{3} \sin 3m$$

or
$$\theta = m + 2e \sin m - 4e^3\left(\frac{3 \sin m - \sin 3m}{4}\right) + 2e^2 \sin 2m$$
$$+ \tfrac{5}{4}e^3(\sin 3m + \sin m) - \tfrac{3}{4}e^2 \sin 2m$$
$$-3e^3\left(\frac{\sin 3m - \sin m}{2}\right) + \frac{e^3}{3} \sin 3m.$$

$$\therefore \quad \theta=m+\left(2e-3e^3+\frac{5e^3}{4}+\tfrac{3}{2}e^3\right)\sin m+\left(2e^2-\frac{3}{4}e^2\right)\sin 2m$$
$$+\left(e^3+\frac{5}{4}e^3-\frac{3e^3}{2}+\frac{e^3}{3}\right)\sin 3m$$

or $\theta=m+(2e-\frac{1}{4}e^3)\sin m+\frac{5}{4}e^2\sin 2m+\frac{13}{12}e^3\sin 3m.$

Ex. 1. *Prove that if e^3 is neglected, then*

$$r=a\,(1-e\cos nt+e^2\sin^2 nt)$$
$$\omega=n\,(1+\tfrac{1}{2}e^2\cos 2nt)$$

where ω is the angular velocity about the empty focus.

(Agra 63)

We know that $r=SP=$focal distance of $P=a-ex$ where (x, y) are the co-ordinates of P referred to C as origin.

$$\therefore \quad x=a\cos\phi.$$

$$\therefore \quad r=a-ae\cos\phi=a\,(1-e\cos\phi). \qquad \text{...(1)}$$

Now we have to introduce nt *i.e.* m in place of ϕ in R. H. S. We know from relation (b) that $\phi=m+e\sin m.$

$$\therefore \quad \cos\phi=\cos(m+e\sin m)$$
$$=\cos m\cos(e\sin m)-\sin m\sin(e\sin m).$$

$$\therefore \quad e\cos\phi=e\,[\cos m\,(1-\ldots)-\sin m\,(e\sin m)],$$
$$e\cos\phi=e\cos m-e^2\sin^2 m.$$

Putting for $e\cos\phi$ in (1), we get

$$r=a\,(1-e\cos m+e^2\sin^2 m)\ \text{where}\ m=nt$$

or
$$r=a\left[1-e\cos m+\frac{e^2}{2}(1-\cos 2m)\right]$$
$$=a\,[1-e\cos m+e^2\sin^2 m].\quad \text{Now put } m=nt.$$

2nd Part. We know that in a central orbit,

$$r^2\frac{d\theta}{dt}=h=vp \text{ about the focus } S.$$

Similarly, $r'^2\dfrac{d\theta'}{dt}=h=v.p'$ about the empty focus.

$$\therefore \quad \frac{d\theta'}{dt}=\frac{v.p'}{r'^2}=\frac{v}{r'}.\frac{p'}{r'}=\frac{v}{r'}.\frac{p}{r}=\frac{h}{rr'},$$

$\because \quad \dfrac{p}{r}=\dfrac{p'}{r'}=$sin (angle between tangent and radius vector).

Again $h=nab$ and $rr'=(a-ex)(a+ex)=a^2-e^2x^2$
or $rr'=a^2-e^2a^2\cos^2\phi$ where (x, y) are the co-ordinates of P referred to C as origin.

$$\therefore\quad \frac{d\theta'}{dt}=\frac{nab}{a^2-e^2.a^2\cos^2\phi}=\frac{nab}{a^2}(1-e^2\cos^2\phi)^{-1}$$

or
$$\frac{d\theta'}{dt}=n.\sqrt{(1-e^2)}\,[1+e^2\cos^2\phi-...]$$
$$=n\,(1-\tfrac{1}{2}e^2...)\,(1+e^2\cos^2\phi+...)$$
$$=n\left[1+\frac{e^2}{2}(2\cos^2\phi-1)...\right]$$
$$=n\left[1+\frac{e^2}{2}\cos 2\phi\right].$$

Now $\phi=m+e\sin m$; $\therefore\ \frac{e^2}{2}\cos 2\phi=\frac{e^2}{2}\cos 2m$ upto e^2.

$$\therefore\quad \frac{d\theta'}{dt}=n\left[1+\frac{e^2}{2}\cos 2m\right]=n\left[1+\frac{e^2}{2}\cos 2nt\right].\ \text{Proved.}$$

Also integrating, we get

$$\theta'=n\left[t+\frac{e^2}{4n}\sin 2nt\right]=nt+\frac{e^2}{4}\sin 2nt.$$

§ 12. Lambert's Theorem.

To prove that time of describing an arc PQ of an elliptic orbit is given by $\frac{2\pi t}{T}=\eta-\eta'-(\sin\eta-\sin\eta')$

where $\sin\frac{\eta}{2}=\frac{1}{2}\sqrt{\left(\frac{r+r'+k}{a}\right)}$, $\sin\frac{\eta'}{2}=\frac{1}{2}\sqrt{\left(\frac{r+r'-k}{a}\right)}$

where T is the periodic time, r, r' being radii vectors SP and SQ and k the length of chord PQ.

Let A be the perihelion position and eccentric angles of P, Q be ϕ and ϕ' and corresponding times be t_1 and t_2 so that the required time t is t_2-t_1.

Now we know from Kepler's equation that

$$m=\phi-e\sin\phi$$

or
$$nt_1=\phi-e\sin\phi,$$
$$nt_2=\phi'-e\sin\phi'.$$

$$\therefore\quad nt=n\,(t_2-t_1)=\phi'-\phi-e\,(\sin\phi'-\sin\phi)$$

or $$nt=\phi'-\phi-e.2\sin\frac{\phi'-\phi}{2}\cos\frac{\phi'+\phi}{2}. \qquad \ldots(1)$$

Put $\phi'-\phi=2\alpha$ and $e\cos\dfrac{\phi'+\phi}{2}=\cos\beta$ and $n=\dfrac{2\pi}{T}$.

$$\therefore\quad \frac{2\pi}{T}.t=2\alpha-2\sin\alpha\cos\beta$$
$$=(\beta+\alpha)-(\beta-\alpha)-[\sin(\beta+\alpha)-\sin(\beta-\alpha)]$$
$$=\eta-\eta'-(\sin\eta-\sin\eta'),$$

where $\eta=\beta+\alpha$, $\eta'=\beta-\alpha$.

Now $$r=a-ex=a(1-e\cos\phi),$$
$$r'=a-ex'=a(1-e\cos\phi').$$

$$\therefore\quad r+r'=2a-ae.2\cos\frac{\phi+\phi'}{2}\cos\frac{\phi'-\phi}{2}$$

or $$r+r'=2a(1-\cos\alpha\cos\beta) \qquad \ldots(2)$$

Again $k^2=PQ^2=a^2(\cos\phi'-\cos\phi)^2+b^2(\sin\phi'-\sin\phi)^2$

or $$k^2=a^2.4\sin^2\frac{\phi+\phi'}{2}\sin^2\frac{\phi-\phi'}{2}$$
$$+a^2(1-e^2).4\cos^2\frac{\phi+\phi'}{2}\sin^2\frac{\phi'-\phi}{2}$$

$$=4a^2\sin^2\alpha\left[\sin^2\frac{\phi+\phi'}{2}+(1-e^2)\cos^2\frac{\phi+\phi'}{2}\right]$$

$$=4a^2\sin^2\alpha\left[1-e^2\cos^2\frac{\phi+\phi'}{2}\right]=4a^2\sin^2\alpha\,[1-\cos^2\beta]$$

$$=4a^2\sin^2\alpha\sin^2\beta.$$

$$\therefore\quad k=2a\sin\alpha\sin\beta. \qquad \ldots(3)$$

$$\therefore\quad r+r'+k=2a\,[1-\cos\alpha\cos\beta+\sin\alpha\sin\beta]$$
$$=2a\,[1-\cos(\alpha+\beta)]$$
$$=4a\sin^2\frac{\alpha+\beta}{2}=4a\sin^2\frac{\eta}{2}$$

or $$\frac{1}{2}\sqrt{\left(\frac{r+r'+k}{a}\right)}=\sin\frac{\eta}{2}.$$

Similary $\dfrac{1}{2}\sqrt{\left(\dfrac{r+r'-k}{a}\right)}=\sin\dfrac{\eta'}{2}.$ Hence proved.

§ 3. Euler's Theorem.

Prove that the time of describing a parabolic orbit is

$$t=\frac{T_0}{12\pi}\left[\left(\frac{r+r'+k}{a_0}\right)^{3/2}-\left(\frac{r+r'-k}{a_0}\right)^{3/2}\right]$$

where $4a_0$ is the latus rectum, T_0 the period of an elliptic orbit of major axis $2a_0$.

We have proved in Lambert's Theorem that the period of an elliptic orbit of major axis $2a$ is given by

$$\frac{2\pi t}{T}=\eta-\eta'-(\sin\eta-\sin\eta')$$

$$=(\eta-\sin\eta)-(\eta'-\sin\eta') \qquad \ldots(1)$$

where $\sin\frac{\eta}{2}=\frac{1}{2}\sqrt{\left(\frac{r+r'+k}{a}\right)}$, $\sin\frac{\eta'}{2}=\frac{1}{2}\sqrt{\left(\frac{r+r'-k}{a}\right)}$ $\quad\ldots(2)$

where k is the length of chord PQ.

Now if the semi-major axis a of the elliptic orbit tends to infinity, then the orbit tends to be parabolic. Hence from (2),

$$\sin\frac{\eta}{2}\to 0 \text{ and } \sin\frac{\eta'}{2}\to 0 \text{ as } a\to\infty.$$

Therefore both η and η' are small.

$$\therefore\quad \eta-\sin\eta=\eta-\left(\eta-\frac{1}{3!}\eta^3\ldots\right)=\frac{\eta^3}{6}$$

and

$$\eta'-\sin\eta'=\eta'-\left(\eta'-\frac{1}{3!}\eta'^3\right)=\frac{\eta'^3}{6}. \qquad \ldots(3)$$

Also we know that for any two elliptic orbits,

$$\frac{T^2}{a^3}=\frac{T_0^2}{a_0^3};\quad\therefore\quad T=T_0\frac{a^{3/2}}{a_0^{3/2}}. \qquad \ldots(4)$$

Also

$$\sin\frac{\eta}{2}=\frac{\eta}{2}=\frac{1}{2}\sqrt{\left(\frac{r+r'+k}{a}\right)}.$$

$$\therefore\quad \eta=\sqrt{\left(\frac{r+r'+k}{a}\right)} \text{ etc.} \qquad \ldots(5)$$

Hence from (1) by the help of (3) and (4), we get

$$\frac{2\pi t}{T_0}\cdot\frac{a_0^{3/2}}{a^{3/2}}=\frac{\eta^3}{6}-\frac{\eta'^3}{6}.$$

$$\therefore \quad t=\frac{T_0}{2\pi}\cdot\frac{a^{3/2}}{a_0^{3/2}}\cdot\frac{1}{6}\left[\left(\frac{r+r'+k}{a}\right)^{3/2}-\left(\frac{r+r'-k}{a}\right)^{3/2}\right]$$

or

$$t=\frac{T_0}{12\pi}\left[\left(\frac{a}{a_0}\cdot\frac{r+r'+k}{a}\right)^{3/2}-\left(\frac{a}{a_0}\cdot\frac{r+r'-k}{a}\right)^{3/2}\right]$$

$$=\frac{T_0}{12\pi}\left[\left(\frac{r+r'+k}{a_0}\right)^{3/2}-\left(\frac{r+r'-k}{a_0}\right)^{3/2}\right].$$

§ 4. **Time for a hyperbolic orbit.**

$$\sqrt{\left(\frac{\mu}{a^3}\right)}\, t=-\eta+\eta'+\sinh\eta-\sinh\eta',$$

where

$$\sinh\frac{\eta}{2}=\frac{1}{2}\sqrt{\left(\frac{r+r'+k}{a}\right)},$$

$$\sinh\frac{\eta'}{2}=\frac{1}{2}\sqrt{\left(\frac{r+r'-k}{a}\right)}$$

and k the length of chord PQ.

We know that $\sqrt{(\mu l)}=h=r^2\frac{d\theta}{dt}=x\frac{dy}{dt}-y\frac{dx}{dt}$.

Now (x, y) are the co-ordinates of P referred to foc as origin.

$$\therefore \quad x=ae-a\cosh\phi,\ y=b\sinh\phi,$$

where $(a\cosh\phi, b\sinh\phi)$ are the co-ordinates of P refer to centre as origin.

$$\therefore \quad \sqrt{(\mu l)}=(ae-a\cosh\phi)\, b\cosh\phi\cdot\frac{d\phi}{dt}$$

$$-b\sinh\phi\cdot(-a\sinh\phi)\frac{d\phi}{dt}$$

$$=ab\,(e\cosh\phi-1)\frac{d\phi}{dt}$$

or

$$\sqrt{\mu}\cdot\frac{b}{\sqrt{a}}\cdot\frac{1}{ab}\,dt=(e\cosh\phi-1)\,d\phi$$

or

$$\sqrt{\left(\frac{\mu}{a^3}\right)}\, t=e\sinh\phi-\phi.$$

$$\therefore \quad \sqrt{\left(\frac{\mu}{a^3}\right)}\, t_1=e\sinh\phi-\phi$$

or

$$\sqrt{\left(\frac{\mu}{a^3}\right)}\, t_2=e\sinh\phi'-\phi'$$

or $\sqrt{\left(\frac{\mu}{a^3}\right)}\,(t_1-t_2)=-(\phi-\phi')+e\,(\sinh\phi-\sinh\phi')$

or $\sqrt{\left(\frac{\mu}{a^3}\right)}\,t=-(\phi-\phi')+2e.\cosh\frac{\phi+\phi'}{2}\sinh\frac{\phi-\phi'}{2}.$

Put $\phi-\phi'=2\alpha$ and $e\cosh\frac{\phi+\phi'}{2}=\cosh\beta.$

$$\therefore\ \sqrt{\left(\frac{\mu}{a^3}\right)}\,t=-2\alpha+2\cosh\beta\sinh\alpha$$
$$=-(\alpha+\beta)+(\beta-\alpha)+\sinh(\alpha+\beta)-\sinh(\beta-\alpha)$$
$$=-\eta+\eta'+\sinh\eta-\sinh\eta',$$

where $\alpha+\beta=\eta$ and $\beta-\alpha=\eta'$.

$r=SP=ex-a$ for a hyperbola $=a\,(e\cosh\phi-1)$,

$r'=a\,(e\cosh\phi'-1)$.

$$\therefore\ r+r'=2ae\cosh\frac{\phi+\phi'}{2}\cosh\frac{\phi-\phi'}{2}-2a$$
$$=2a\,(\cosh\beta\cosh\alpha-1).$$

$k^2=(\text{chord } PQ)^2=a^2\,[\cosh\phi-\cosh\phi']^2+b^2\,[\sinh\phi-\sinh\phi']^2$

$$=4a^2\sinh^2\frac{\phi+\phi'}{2}\sinh^2\frac{\phi-\phi'}{2}$$
$$+a^2\,(e^2-1)\,4\sinh^2\frac{\phi-\phi'}{2}\cosh^2\frac{\phi+\phi'}{2}$$
$$=4a^2\sinh^2\alpha\left[e^2\cosh^2\frac{\phi+\phi'}{2}-1\right]$$
$$=4a^2\sinh^2\alpha\,[\cosh^2\beta-1]=4a^2\sinh^2\alpha\sinh^2\beta.$$

$k=2a\sinh\alpha\sinh\beta.$

$$\therefore\ r+r'+k=2a\,[\cosh\beta\cosh\alpha+\sinh\alpha\sinh\beta-1]$$
$$=2a\,[\cosh(\alpha+\beta)-1]=4a\sinh^2\frac{\alpha+\beta}{2}\,..$$

$$\therefore\ r+r'+k=4a\sinh^2\frac{\eta}{2}\,.$$

Similarly $\qquad r+r'-k=4a\sinh^2\frac{\eta'}{2}\,.$

CHAPTER III

CONSTRAINED MOTION

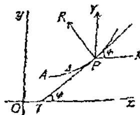
§ 1. *A particle is moving on a smooth plane curve under the action of conservative system of forces in the plane; to discuss the motion and prove the following :*

(a) Under the action of conservative system of forces the change in kinetic energy of the particle is equal to the work done by the forces. This is known as equation of energy.

(b) Under the action of conservative system of forces the sum of the kinetic and potential energies of the particle is constant throughout the motion. This is known as principle of conservation of energy.

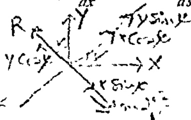
(Cal. Hon's. 63, 61 ; Agra 60 ; Punjab 61)

Let $P(x, y)$ be the position of the particle at any time t at a distance s measured along the curve from a fixed point A on the curve.



Again suppose that X and Y are the components of the external forces acting on the particle parallel to the axes of co-ordinates and R be the normal reaction of the curve acting through P the point of contact. The velocity at P may be taken to be v and the inclination of tangent at P to the axis of x be denoted by ψ ; then we know that

$$\tan \psi = \frac{dy}{dx} \text{ and } \cos \psi = \frac{dx}{ds} \text{ and } \sin \psi = \frac{dy}{ds}. \quad \ldots(1)$$



Resolving along the tangent and normal, we get by using $mf=P$.

$$m\frac{dv}{ds}=X\cos\psi+Y\sin\psi, \qquad ...(2)$$

$$m\frac{v^2}{\rho}=R-X\sin\psi+Y\cos\psi. \qquad ...(3)$$

From (2) by the help of (1), we have

$$mv\frac{dv}{ds}=X\frac{dx}{ds}+Y\frac{dy}{ds}\ ; \quad \therefore \quad mv\,dv=X\,dx+Y\,dy.$$

Integrating, we get $\frac{1}{2}mv^2=\int(X\,dx+Y\,dy)$.

Now let us suppose that $X\,dx+Y\,dy$ is complete differential of some function of x, y say $f(x, y)$; then we have

$$\tfrac{1}{2}mv^2=f(x, y)+C. \qquad ...(4)$$

where C is a constant of integration whose value is to be determined by intial conditions. Above gives us the K. E. of the particle. Let us suppose that the particle is projected from the point (x_0, y_0) with velocity u so that from (4), we get

$$\tfrac{1}{2}mu^2=f(x_0, y_0)+C. \qquad ...(5)$$

Subtracting (4) and (5), we get

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mu^2=f(x, y)-f(x_0, y_0). \qquad ...(6)$$

The above result depends anly on the initial and final positions *i.e.* co-ordinates of the particle and is quite independent of the path pursued by it, *i.e.* independent of the form of the restraining curve.

The function $f(x, y)=\int(X\,dx+Y\,dy)$ is called work function and is equal to the work done on the particle by the external forces. Hence we can interpret equation (6) as under :

L. H. S.=change in the K. E. of the particle as it moves from initial position to final position.

R. H. S.=work done by external forces in displacing the particle from its initial position (x_0, y_0) to the position $P(x, y)$.

$\therefore$ change in K. E. = work done.

Note. When the work done by the forces in moving the particle from one position to another depends only upon its initial and final positions, the system of forces is called conservative system.

Potential Energy. The potential energy of the particle when it is at some point (x, y) is given by the amount of work done by the forces as the particle moves from this position to some standard position say (x', y').

$$\therefore \quad \text{P. E.} = \int_{(x,\, y)}^{(x',\, y')} (X\,dx + Y\,dy)$$

$$= \Big[f(x, y) \Big]_{x\ y}^{x',\ y'} = f(x', y') - f(x, y). \quad \ldots(7)$$

$$\text{K. E.} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}mu^2 + f(x, y) - f(x_0, y_0) \text{ by } (6) \ldots(8)$$

Adding (7) and (8) we get

$$\text{P.E.} + \text{K.E.} = f(x', y') - f(x, y) + \tfrac{1}{2}mu^2 + f(x, y) - f(x_0, y_0)$$

$$= \tfrac{1}{2}mu^2 + f(x', y') - f(x_0, y_0) \text{ i.e. constant}$$

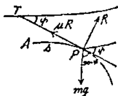
Hence we have **K. E. + P.E. = constant.** **(Punjab 56, 60)**

Having found the value of v^2 from (6) and putting in (3) we shall get the value of normal reaction R.

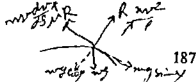
§ 2. *Motion on a rough curve under gravity.*

A particle slides down a rough curve under gravity in a vertical plane ; to discuss the motion

(Sagar 62 ; Agra 49, 59, 63, 64 ; Punjab 57)



Let P be the position of particle at any time t at an arcural distance s measured from a fixed point A. The tangent PT at P is inclined to the horizontal at an angle ψ. R is normal reaction acting as shown and μR is the force of friction acting upwards along the tangent as the particle is sliding downwards.



Resolving along the tangent and normal, we have by using $P=mf$ the following equations of motion

$$mv\frac{dv}{ds}=mg\sin\psi-\mu R \qquad \ldots(1)$$

and

$$m\frac{v^2}{\rho}=mg\cos\psi-R. \qquad \ldots(2)$$

Equation (1) can be written as

$$\tfrac{1}{2}m\frac{dv^2}{ds}=mg\sin\psi-\mu R. \qquad \ldots(3)$$

Multiplying (2) by μ and subtracting from (3) thereby eliminating R, we get

$$\tfrac{1}{2}m\frac{dv^2}{ds}-\mu m\frac{v^2}{\rho}=mg(\sin\psi-\mu\cos\psi)$$

or
$$\frac{dv^2}{ds}\cdot\rho-2\mu v^2=2g\rho(\sin\psi-\mu\cos\psi)$$

or
$$\frac{dv^2}{ds}\cdot\frac{ds}{d\psi}-2\mu v^2=2g\rho(\sin\psi-\mu\cos\psi)$$

or
$$\frac{dv^2}{d\psi}-2\mu v^2=2g\rho(\sin\psi-\mu\cos\psi).$$

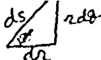
Above is a linear differential equation in which v^2 is dependent and ψ is independent variable.

Clearly I. F. $=e^{\int P\,d\psi}=e^{\int -2\mu\,d\psi}=e^{-2\mu\psi}$.

Multiplying both sides by $e^{-2\mu\psi}$ and intergrating, we get

$$v^2.e^{-2\mu\psi}=2g\int\rho.e^{-2\mu\psi}(\sin\psi-\mu\cos\psi)\,d\psi+C. \qquad \ldots(4)$$

Now when the equation of the curve be given, then ρ can be found in terms of ψ and we can integrate the R. H. S. The value of constant of integration C is to be determined by applying initial conditions. Thus we shall find the value of v^2 and having found v^2 and putting in (2), we shall get the value of R.



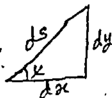
§ 3. Certain Important formulae to be remembered.

1. $\tan\phi = r\dfrac{d\theta}{dr}$, $\sin\phi = r\dfrac{d\theta}{ds}$, $\cos\phi = \dfrac{dr}{ds}$.

2. Pedal equation of equiangular spiral $r = ae^{\theta\cot\alpha}$ is
$$p = r\sin\alpha.$$

3. Radius of curvature
$$\rho = \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}$$
$$= r\frac{dr}{dp} = r\bigg/\frac{dp}{dr}.$$

4. $\tan\psi = \dfrac{dy}{dx}$, $\cos\psi = \dfrac{dx}{ds}$, $\sin\psi = \dfrac{dy}{ds}$.



5. $\dfrac{ds}{d\theta} = \sqrt{\left[r^2+\left(\dfrac{dr}{d\theta}\right)^2\right]}$.

6. In the ellipse $x = a\cos\phi$, $y = b\sin\phi$,
$$\rho = \frac{(a^2\sin^2\phi + b^2\cos^2\phi)^{3/2}}{ab} = \frac{CD^3}{ab},$$

where CD is the semi-conjugate diameter.

7. $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx),$$
$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx).$$

Exercise

Ex. 1. (a) *A particle slides down the smooth curve $y = a\sinh\dfrac{x}{a}$ the axis of x being horizontal and axis of y downwards startting from rest at the point where the tangent is inclined at an angle α to the horizon. Show that it will leave the curve when it has fallen through a vertical distance $a\sec\alpha$*

Refer fig. § 2. There will be no force of friction μR as the curve is smooth.

The equations of motion are

$$mv\frac{dv}{ds}=mg\sin\psi \text{ and } m\frac{v^2}{\rho}=mg\cos\psi-R.$$

$$\therefore \quad v\frac{dv}{ds}=g\frac{dy}{ds} \text{ [§ 3] ;} \quad \therefore \quad v\,dv=g\,dy.$$

Integrating, we get $v^2=2gy+A$.

Initially when $y=y_0$ say $v=0$, $\therefore A=-2gy_0$.

$$\therefore \quad v^2=2g(y-y_0). \quad \text{...(1)}$$

Putting for v^2 in the other equation of motion and also if the particle leaves, then $R=0$.

$$\therefore \quad m.2g(y-y_0)=mg\cos\psi.\rho. \quad \text{.. (2)}$$

Now $y=a\sinh\dfrac{x}{a}$; $\therefore \dfrac{dy}{dx}=\cosh\dfrac{x}{a}=\tan\psi=y'$

or $$\tan\psi=\sqrt{\left(1+\sinh^2\frac{x}{a}\right)}=\sqrt{\left(1+\frac{y^2}{a^2}\right)}.$$

Initially when $\psi=\alpha$, $y=y_0$.

$$\therefore \quad \tan\alpha=\sqrt{\left(1+\frac{y_0^2}{a^2}\right)}. \quad \text{...(3)}$$

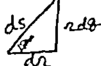
$$\rho\cos\psi=\frac{(1+y'^2)^{3/2}}{y''}\cdot\frac{1}{\sqrt{(1+y'^2)}}=\frac{(1+y'^2)}{y''}=a\frac{\left(1+\cosh^2\frac{x}{a}\right)}{\sinh\frac{x}{a}}$$

or $$\rho\cos\psi=a\frac{1+1+\sinh^2\frac{x}{a}}{\sinh\frac{x}{a}}=a^2\frac{\left(2+\frac{y^2}{a^2}\right)}{y}=\frac{2a^2+y^2}{y}.$$

Hence from (2), we get

$$2(y-y_0)=\frac{2a^2+y^2}{y} \quad \text{or} \quad 2y^2-2yy_0=2a^2+y^2$$

or $$y^2-2yy_0+y_0^2=2a^2+y_0^2$$



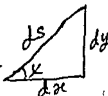
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4. $\tan\psi = \frac{dy}{dx}$, $\cos\psi = \frac{dx}{ds}$, $\sin\psi = \frac{dy}{ds}$.



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where CD is the semi-conjugate diameter.

7. $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx),$$
$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx).$$

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Ex. 1. (a) *A particle slides down the smooth curve* $y = a\sinh\frac{x}{a}$ *the axis of x being horizontal and axis of y downwards startting from rest at the point where the tangent is inclined at an angle* α *to the horizon. Show that it will leave the curve when it has fallen through a vertical distance* $a\sec\alpha$

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Integrating, we get $v^2=2gy+A$.

Initially when $y=y_0$ say $v=0$, $\therefore\ A=-2gy_0$.

$$\therefore\quad v^2=2g\,(y-y_0). \qquad \text{...(1)}$$

Putting for v^2 in the other equation of motion and also if the particle leaves, then $R=0$.

$$\therefore\quad m.2g\,(y-y_0)=mg\cos\psi\,.\,\rho. \qquad \text{...(2)}$$

Now $y=a\sinh\frac{x}{a}$; $\therefore\ \frac{dy}{dx}=\cosh\frac{x}{a}=\tan\psi=y'$

or
$$\tan\psi=\sqrt{\left(1+\sinh^2\frac{x}{a}\right)}=\sqrt{\left(1+\frac{y^2}{a^2}\right)}.$$

Initially when $\psi=\alpha$, $y=y_0$.

$$\therefore\quad \tan\alpha=\sqrt{\left(1+\frac{y_0^2}{a^2}\right)}. \qquad \text{...(3)}$$

$$\rho\cos\psi=\frac{(1+y'^2)^{3/2}}{y''}\cdot\frac{1}{\sqrt{(1+y'^2)}}=\frac{(1+y'^2)}{y''}=a\,\frac{\left(1+\cosh^2\frac{x}{a}\right)}{\sinh\frac{x}{a}}$$

or
$$\rho\cos\psi=a\,\frac{1+1+\sinh^2\frac{x}{a}}{\sinh\frac{x}{a}}=a^2\,\frac{\left(2+\frac{y^2}{a^2}\right)}{y}=\frac{2a^2+y^2}{y}.$$

Hence from (2), we get

$$2\,(y-y_0)=\frac{2a^2+y^2}{y}\quad\text{or}\quad 2y^2-2yy_0=2a^2+y^2$$

or
$$y^2-2yy_0+y_0^2=2a^2+y_0^2$$

In order that the particle may move comple the circle its velocity v should not vanish till the p reached the other end of the diameter *i.e.* B where $\to 0$. Hence the least velocity of projection is obtained by putting $v=0$ and $r=a+b$ in (3).

$$\therefore \quad 0-\frac{1}{2}V^2=\frac{\mu}{a+b}-\frac{\mu}{a-b}=-\frac{2\mu b}{a^2-b^2}.$$

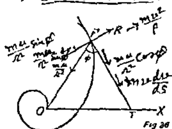
$$\therefore \quad V^2=\frac{4\mu b}{a^2-b^2} \text{ or } V=\sqrt{\left(\frac{4\mu b}{a^2-b^2}\right)}.$$

Ex. 2. (a) *A bead is constrained to move on a smooth wire in the form of an equiangular spiral. It is attached to the pole of the spiral by a force* $m\mu$ *(distance)*$^{-2}$ *and starts from rest at a distance b from the pole. Show that if the equation of the spiral be* $r=ae^{\theta \cot \alpha}$, *the time of arriving at the pole is*

$$\frac{\pi}{2}\sqrt{\left(\frac{b^2}{2\mu}\right)} \sec \alpha.$$

Find also the reaction of the curve at any instant.

(Punjab 69 ; Rajputana 62 ; Agra 66, 64, 50, 46)



Just as in last question, we have the tangential equation of motion as

$$mv\frac{dv}{ds}=-\frac{m\mu}{r^2}\cos\phi=-\frac{m\mu}{r^2}\cdot\frac{dr}{ds}.$$

$$\therefore \quad \int v\,dv=\int -\frac{\mu}{r^2}\,dr$$

or

$$\frac{1}{2}v^2=\frac{\mu}{r}+A.$$

Initially when $r=b$, $v=0$.

$$\therefore \quad 0=\frac{\mu}{b}+A \quad \text{or} \quad A=-\frac{\mu}{b}.$$

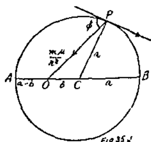
$$\therefore \quad v^2=2\mu\left(\frac{1}{r}-\frac{1}{b}\right). \qquad \text{...(1)}$$

or $\quad (y-y_0)^2=2a^2+a^2(\tan^2\alpha-1)=a^2(1+\tan^2\alpha)$ by (3)

or $\quad (y-y_0)^2=a^2\sec^2\alpha\ ;\quad \therefore\quad y-y_0=a\sec\alpha.$

Hence the particle will leave when it has fallen through a vertical distance $a\sec\alpha$.

Ex. 1. (b) *A small bead, of mass m, moves on a smooth circular wire, being acted upon by a central attraction $\dfrac{m\mu}{(distance)^2}$ to a point within the circle situated at a distance b from the centre. Show that in order that the bead may move completely round the circle, its velocity of projection at the point of the wire nearest the centre of force must not be less than $\sqrt{\left\{\dfrac{4\mu b}{(a^2-b^2)}\right\}}$.* (Vikram 65, Agra 42)



From the figure it is clear that $OA=a-b,\ OB=a+b.$ Let the velocity at A be V and that at P be v. The central attraction is $\dfrac{m\mu}{r^2}$.

The equation of motion by $P=mf$ (along the tangent) is

$$mv\frac{dv}{ds}=-\frac{m\mu}{r^2}\cos\phi=-\frac{m\mu}{r^2}\cdot\frac{dr}{ds}.$$

$$\therefore\quad \int v\,dv=\int -\frac{\mu}{r^2}\,dr\quad\text{or}\quad \frac{1}{2}v^2=\frac{\mu}{r}+A. \qquad ...(1)$$

When the particle is at A where $r=a-b$, the velocity is V.

$$\therefore\quad \frac{1}{2}V^2=\frac{\mu}{a-b}+A. \qquad ...(2)$$

Subtracting (1) and (2), $\dfrac{1}{2}v^2-\dfrac{1}{2}V^2=\dfrac{\mu}{r}-\dfrac{\mu}{a-b}.\qquad ...(3)$

In order that the particle may move comple the circle its velocity v should not vanish till the p reached the other end of the diameter *i.e.* B where Hence the least velocity of projection is obtained by putting $v=0$ and $r=a+b$ in (3).

$$\therefore \quad 0-\frac{1}{2}V^2=\frac{\mu}{a+b}-\frac{\mu}{a-b}=-\frac{2\mu b}{a^2-b^2}.$$

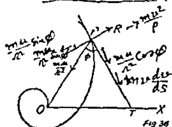
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Initially when $r=b$, $v=0$.

$$\therefore \quad 0=\frac{\mu}{b}+A \quad \text{or} \quad A=-\frac{\mu}{b}.$$

$$\therefore \quad v^2=2\mu\left(\frac{1}{r}-\frac{1}{b}\right). \qquad \ldots(1)$$

$$v=\frac{ds}{dt}=-\sqrt{(2\mu)}\sqrt{\left(\frac{b-r}{br}\right)}$$

(s decreasing, $\therefore$ —ive sign).

$$\therefore \quad \frac{ds}{dr}\cdot\frac{dr}{dt}=-\sqrt{(2\mu)}\sqrt{\left(\frac{b-r}{br}\right)}.$$

Now $p=r\sin\alpha=r\sin\phi$ is the pedal equation of spiral.

$$\therefore \quad \phi=\alpha\ ; \quad \therefore \quad \frac{dr}{ds}=\cos\phi=\cos\alpha \quad \text{or} \quad \frac{ds}{dr}=\sec\alpha.$$

$$\therefore \quad \sec\alpha.\frac{dr}{dt}=-\sqrt{(2\mu)}\sqrt{\left(\frac{b-r}{br}\right)}$$

or $$\int_{t=0}^{T}\sqrt{(2\mu)}.\cos\alpha.dt=\int_{r=b}^{0}-\sqrt{\left(\frac{br}{b-r}\right)}\,dr.$$

The limits are chosen as when $r=b$, $t=0$ and when $r=0$ *i.e.* the particle arrives at pole the time is supposed to be T.

Put $r=b\sin^2\theta$; $\therefore$ $dr=2b\sin\theta\cos\theta\,d\theta$.

Also when $r=0$, $\theta=0$ and when $r=b$, $\theta=\frac{\pi}{2}$.

$$\therefore \quad \sqrt{(2\mu)}\cos\alpha.\Big[\,t\,\Big]_0^T=-\int_{\pi/2}^{0}\sqrt{\left(\frac{b^2\sin^2\theta}{b\cos^2\theta}\right)}\times 2b\sin\theta\cos\theta\,d\theta.$$

$$\therefore \quad \sqrt{(2\mu)}\cos\alpha.T=\int_0^{\pi/2}2b^{3/2}.\sin^2\theta\,d\theta=2b^{3/2}.\frac{1}{2}.\frac{\pi}{2}.$$

$$\therefore \quad T=\frac{\pi}{2}\sqrt{\left(\frac{b^3}{2\mu}\right)}.\sec\alpha.$$

Again in order to find the reaction we write the equation of motion along the normal.

$$m\frac{v^2}{\rho}=\frac{m\mu}{r^2}\sin\phi-R.$$

Now $\phi=\alpha$. Also $\rho=r.\frac{dr}{dp}=\frac{r}{\frac{dp}{dr}}=\frac{r}{\sin\alpha}$. $\because$ $p=r\sin\alpha$

$\int_0^{\pi/2} 2\sin^2\theta\,d\theta=\frac{1}{2}\int_0^{\pi/2}[1-\cos 2\theta]\,d\theta$

and $\quad v^2=2\mu\left(\frac{1}{r}-\frac{1}{b}\right)$ by (1).

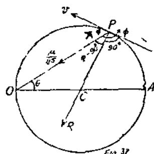
$$\therefore \quad R=\frac{m\mu}{r^2}\sin\phi-\frac{mv^2}{\rho}$$

or
$$R=\frac{m\mu}{r^2}\sin\alpha-\frac{m}{r}\sin\alpha\,.\,2\mu\left(\frac{1}{r}-\frac{1}{b}\right).$$

$$=\frac{m\mu\sin\alpha}{r}\left[\frac{1}{r}-\frac{2}{r}+\frac{2}{b}\right]=\frac{m\mu\sin\alpha}{r}\left[\frac{2}{b}-\frac{1}{r}\right].$$

Ex. 2. (b) *A small bead is projected with any velocity along a smooth circular wire under the action of a force varying inversely as the fifth power of the distance from a centre of force situated on the circumference. Prove that the pressure on the wire is constant.*

Let the diameter $OA=2a$ coincide with the initial line. The equation of the circle is $r=2a\cos\theta$. Let us choose O as the centre of force and R be the pressure. We have the following equation :



$$v\frac{dv}{ds}=-\frac{\mu}{r^5}\cos\phi=-\frac{\mu}{r^5}\frac{dr}{ds}.$$

$$\therefore \quad v\,dv=-\frac{\mu}{r^5}dr.$$

Integrating, $\qquad v^2=\frac{\mu}{2r^4}+A.$

Initially at A, $r=2a$ and suppose that $v=u$.

$$\therefore \quad A=u^2-\frac{\mu}{32a^4}.$$

$$\therefore \quad v^2=\frac{\mu}{2r^4}+u^2-\frac{\mu}{32a^4}. \qquad \ldots(1)$$

The other equation of motion along the normal is

$$\frac{v^2}{\rho}=R+\frac{\mu}{r^5}\cos(\phi-90)=R+\frac{\mu}{r^5}\sin\phi.$$

$\rightarrow = \frac{1}{2}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\pi/2}=\frac{1}{2}.\frac{\pi}{2}-0.$

$\because \sin\pi=\sin 0=0.$

or $$\frac{v^2}{a}=R+\frac{\mu}{r^5}\cdot r\frac{d\theta}{ds}, \quad \because \quad \rho=a \text{ and } \sin\phi=r\frac{d\theta}{ds}.$$

Now $r=2a\cos\theta$; $\therefore$ $\frac{dr}{d\theta}=-2a\sin\theta$.

$$\therefore \quad \frac{ds}{d\theta}=\sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}=2a.$$

$$\therefore \quad \frac{v^2}{a}-\frac{\mu}{r^4}\cdot\frac{1}{2a}=R$$

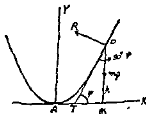
or $$\frac{1}{a}\left[\frac{\mu}{2r^4}+u^2-\frac{\mu}{32a^4}\right]-\frac{\mu}{r^4.2a}=R \text{ by (1)}$$

or $$\frac{u^2}{a}-\frac{\mu}{32a^5}=R.$$

Above shows that R is independent of r and θ and hence constant.

Ex. 3. *A smooth parabolic tube is placed vertex downwards in a vertical plane. A particle slides down the tube from rest under the influence of gravity. Prove that in any position the reaction of the tube is $2\omega\frac{h+a}{\rho}$, where ω is the weight of the particle, ρ the radius of curvature, $4a$ the latus rectum, h the original vertical height of the particle above the vertex.*

Tangential equation of motion is



$$mv.\frac{dv}{ds}=-mg\cos(90-\psi)$$
$$=-mg\sin\psi$$

or $$v\frac{dv}{ds}=-g.\frac{dy}{ds}$$

or $$v\,dv=-g\,dy.$$

Integrating, $v^2=-2gy+A$. When $y=h$, $v=0$; $\therefore$ $A=2gh$.

$$\therefore \quad v^2=2g(h-y). \qquad ...(1)$$

In order to find the reaction, we write the normal equation of motion as

$$m\frac{v^2}{\rho}=R-mg\cos\psi.$$

$$\therefore\quad R=mg\cos\psi+\frac{m}{\rho}\,2g\,(h-y)\text{ by (1)}.\qquad ...(2)$$

The equation of the parabola is $x^2=4ay$.

$$\therefore\quad \tan\psi=\frac{dy}{dx}=\frac{x}{2a}=\sqrt{\left(\frac{y}{a}\right)},\qquad \therefore\quad \cos\psi=\sqrt{\left(\frac{a}{y+a}\right)}.$$

Also $\dfrac{d^2y}{dx^2}=\dfrac{1}{2a}$; $\quad\therefore\quad \rho=\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}\div\dfrac{d^2y}{dx^2}$

or

$$\rho=2a\left(\frac{y+a}{a}\right)^{3/2}.$$

Putting for ρ and $\cos\psi$ in (2), we get

$$R=mg\sqrt{\left(\frac{a}{y+a}\right)}+\frac{m}{\rho}\,2g\,(h-y)$$

$$=\frac{mg}{\rho}\sqrt{\left(\frac{a}{y+a}\right)}.2a\left(\frac{y+a}{a}\right)^{3/2}+\frac{m}{\rho}\,2g\,(h-y)$$

$$=\frac{mg}{\rho}\,[2\,(y+a)+2h-2y]=\frac{2w}{\rho}(h+a)\quad\because\quad w=mg.$$

Ex. 4. *A particle of mass m moves in a smooth circular tube of radius a under the action of a force equal to $m\mu\times$ distance to a point inside the tube at a distance c from its centre. If the particle be placed very nearly at its greatest distance from the centre of force, show that it will describe the quadrant ending at its least distance in time*

$$\sqrt{\left(\frac{a}{\mu c}\right)}\log(\sqrt{2}+1).$$

C is the centre of the circle and O is the centre of force such that $OC=c$ (given) and $CA=a$. The particle starts from A at its greatest distance from centre of force O, where $OA=a+c$ and the least distance is at B, where $OB=a-c$. We are to calculate the time of describing a

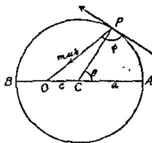


Fig 33

quadrant ending at B *i. e.* where $\theta=\pi$. Hence we have to find the time from $\theta=\frac{\pi}{2}$ to $\theta=\pi$ *i. e.* a quadrant ending at B. Equation of motion of the particle, when at P, where $OP=r$, is

$$mv\frac{dv}{ds}=-m\mu r\cos\phi=-m\mu r.\frac{dr}{ds}.$$

Integrating, we get

$$\left[\frac{v^2}{2}\right]_0^v=-\mu\left[\frac{r^2}{2}\right]_{a+c}^r,$$

as the particle starts from rest at A, where $OA=a+c$.

$$\therefore\quad v^2=\mu\,[(a+c)^2-r^2)\quad\text{or}\quad v=\sqrt{\mu}\sqrt{[(a+c)^2-r^2]}\ \ldots(1)$$

Now in the case of a circle $s=a\theta$, $\therefore\ v=\frac{ds}{dt}=a\,\frac{d\theta}{dt}$.

Also from $\triangle OCP$ we have by cosine formula,

$$OP^2=OC^2+CP^2-2OC.CP\cos(\pi-\theta)$$

or $$r^2=c^2+a^2+2ca\cos\theta$$

or $$r^2=(a+c)^2-2ac\,(1-\cos\theta).$$

$$\therefore\quad (a+c)^2-r^2=2ac\,(1-\cos\theta)=2ac.2\sin^2\frac{\theta}{2}.$$

Hence from (1), we get

$$a\frac{d\theta}{dt}=\sqrt{\mu}.2\sqrt{(ac)}.\sin\frac{\theta}{2}$$

or $$\sqrt{\left(\frac{a}{\mu c}\right)}\int_{\pi/2}^{\pi}\frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\,d\theta=\int_0^T dt$$

or $$\sqrt{\left(\frac{a}{\mu c}\right)}\cdot\left[\log\tan\frac{\theta}{4}\right]_{\pi/2}^{\pi}=T$$

or $$\sqrt{\left(\frac{a}{\mu c}\right)}\left[\log\tan\frac{\pi}{4}-\log\tan\frac{\pi}{8}\right]=T.\qquad\ldots(2)$$

Now $\tan \frac{\pi}{4}=1$; $\therefore$ $\log \tan \frac{\pi}{4}=\log 1=0.$

Again $\tan \frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$; $\therefore$ $\tan \frac{\pi}{8}=\frac{1-\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}}$

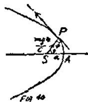
$$=(\sqrt{2}-1)=\frac{2-1}{(\sqrt{2}+1)}.$$

$$\therefore \sqrt{\left(\frac{a}{\mu c}\right)}\,[0-\{-\log(\sqrt{2}+1)\}]=T$$

or
$$T=\sqrt{\left(\frac{a}{\mu c}\right)}\log(\sqrt{2}+1).$$

Ex. 5. *A particle is projected from the vertex of a smooth parabolic tube of latus rectum 4a along the tube and is acted upon by a repulsive force* $\frac{mgr}{c}$ *from the focus. If the velocity of projection is that which would be acquired in moving from focus to the vertex, prove that the time of describing angle* θ *about the focus is* $2\sqrt{\left(\frac{c}{g}\right)}\log\tan\frac{(\pi+\theta)}{4}$.

The polar equation of a parabola of latus rectum $4a$ referred to focus as pole is



$$\frac{2a}{r}=1+\cos\theta$$

or
$$r=\frac{2a}{2\cos^2\frac{\theta}{2}}=a\sec^2\frac{\theta}{2}$$

If V be the velocity from focus to vertex, *i. e.* from $r=0$ to $r=a$ under the given force, then

$$\left[\frac{v^2}{2}\right]_0^V=\frac{g}{c}\left[\frac{r^2}{2}\right]_0^a$$

or $$V^2=\frac{g}{c}\cdot a^2. \qquad \text{...(1)}$$

Above is the velocity of projection.

Writing down the tangential equation of motion, we have

$$mv\frac{dv}{ds}=mg\frac{r}{c}\cos\phi=mg\frac{r}{c}\cdot\frac{dr}{ds} \quad \text{or} \quad v\,dv=\frac{g}{c}r\,dr.$$

Integrating, we get $\left[\frac{v^2}{2}\right]_V^v=\frac{g}{c}\left[\frac{r^2}{2}\right]_a^r$

as the velocity of projection at the vertex $r=a$ is V given.

$$\therefore \quad \frac{v^2}{2}-\frac{V^2}{2}=\frac{g}{c}\left(\frac{r^2}{2}-\frac{a^2}{2}\right)=\frac{gr^2}{2c}-\frac{V^2}{2} \text{ by (1)}.$$

$$\therefore \quad v^2=\frac{g}{c}r^2 \quad \text{or} \quad v=\frac{ds}{dt}=\sqrt{\left(\frac{g}{c}\right)}\,r$$

or $$\frac{ds}{d\theta}\cdot\frac{d\theta}{dt}=\sqrt{\left(\frac{g}{c}\right)}\cdot a\sec^2\frac{\theta}{2}. \qquad \text{...(2)}$$

Now $r=a\sec^2\frac{\theta}{2}$; $\therefore$ $\frac{dr}{d\theta}=2a\sec\frac{\theta}{2}\cdot\sec\frac{\theta}{2}\cdot\tan\frac{\theta}{2}\cdot\frac{1}{2}$.

$$\therefore \quad \frac{ds}{d\theta}=\sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}=a\sec^2\frac{\theta}{2}\sqrt{(1\ [illegible]\ \tan^2\frac{\theta}{2})}=a\sec^3\frac{\theta}{2}.$$

Hence from (2), we [illegible]

$$a\cdot\sec^3\frac{\theta}{2}\Big(\ [illegible] \qquad a\sec^2\frac{\theta}{2} \ [illegible]$$

or $$\sqrt{\left(\frac{c}{g}\right.} \ [illegible] \qquad [illegible]\Big]_0^T \ [illegible]$$

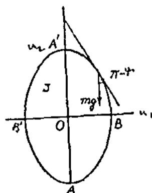
or $$\sqrt{\left(\frac{c}{g}\right)} \ [illegible]$$

or [illegible]

Ex. 6. (a) *A particle is projected hori[zontally from the] lowest point of a smooth elliptic arc whose [major axis is] vertical and moves under gravity along the [curve. Show] that it will leave the curve at some point if the [velocity of] projection lies between* $\sqrt{(2ga)}$ *and* $\sqrt{[ga(5-e^2)]}$ *and if the velocity have the later value, prove that the particle will continue to move round the ellipse in time*

$$2\sqrt{\frac{a}{g}}\int_0^{\pi}\left\{\frac{1-e^2\cos^2\phi}{3-e^2+2\cos\phi}\right\}^{1/2} d\phi. \quad \text{(Rajasthan 63)}$$

Let u be the velocity of projection and u_1 and u_2 be the velocities at B and A' where it will reach after rising a height of a and $2a$ respectively against gravity. Hence by the principle of work and energy, we have



$$\tfrac{1}{2}m(u_1^2-u^2)=-mg.a.$$

$$\therefore \quad u_1^2=u^2-2ga. \quad ...(1)$$

$$\tfrac{1}{2}m(u_2^2-u^2)=-mg.2a.$$

$$\therefore \quad u_2^2=u^2-4ga. \quad ...(2)$$

Now if the particle were to leave the arc it will do so at any point between B and A'. In other words the velocity u_1 at B should be real and +ive *i.e.* $u_1^2>0$ or $u^2-2ga>0$ *i.e* $u^2>2ga$ from (1).

Now suppose R be the reaction at A' then we have the equation of motion as

$$\frac{mv^2}{\rho}=mg+R \text{ or } R=\frac{m}{\rho}(u^2-4ga)-mg \text{ by (2)} \quad ...(3)$$

Now $x=a\cos\phi$, $y=b\sin\phi$ and for the point A, $\phi=0$ and for A', $\phi=\pi$.

$$\rho=\frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}=\frac{1}{ab}[a^2\sin^2\phi+b^2\cos^2\phi]^{3/2}=\frac{1}{ab}.b^3 \text{ for } A'.$$

Putting in (3), we get

$$\therefore \quad R=m\frac{a}{b^2}(u^2-4ag)-mg=m\frac{a}{b^2}u^2-mg\left(4\frac{a^2}{b^2}+1\right). \quad ...(4)$$

$$\tfrac{1}{2}m(v^2-V^2)=-mgh=-mgr\cos\theta$$

or
$$v^2=V^2-2gr\cos\theta. \quad ..(1)$$

Now in order that the particle may make a complete revolution, the reaction at A the highest point should be +ive. Hence we shall calculate the reaction at A, where $\theta=0$.

Normal equation of motion is

$$m\frac{v^2}{\rho}=R-mg\sin(\theta-\phi)$$

or
$$R=\frac{m}{\rho}(V^2-2gr\cos\theta)+mg\sin(\theta-\phi) \text{ by (1)......(2)}$$

Now equation of the lemniscate is $r^2=a^2\cos 2\theta$.

$$\therefore\quad 2\log r=\log a^2+\log\cos 2\theta.$$

$$\therefore\quad 2\cdot\frac{1}{r}\cdot\frac{dr}{d\theta}=\frac{1}{\cos 2\theta}\times-2\sin 2\theta \text{ or } \cot\phi=-\tan 2\theta.$$

$$\therefore\quad \phi=90+2\theta.$$

$$\therefore\quad p=r\sin\phi=r\cos 2\theta=r\cdot\frac{r^2}{a^2}=\frac{r^3}{a^2}.$$

$$\therefore\quad \frac{dp}{dr}=\frac{3r^2}{a^2};\quad\therefore\quad \rho=r\cdot\frac{dr}{dp}=\frac{r}{dp/dr}=r\cdot\frac{a^2}{3r^2}=\frac{a^2}{3r}.$$

Now at the highest point $\theta=0$.

point on the wire distant R from the focus with the velocity which would cause it to describe the ellipse freely under a force $\frac{\mu}{r^2}$. Show that the reaction of the wire is

$$\frac{\lambda}{\rho}\left(\frac{1}{r^2}-\frac{1}{ar}+\frac{1}{R^2}\right)$$

where ρ is the radius of curvature

(Delhi Hons. 63; Agra 44, 57)

We know from last chapter that under inverse square law if the particle is to describe an elliptic orbit, then $v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right)$. If V be the velocity at a point distant R from the focus, then $V^2=\mu\left(\frac{2}{R}-\frac{1}{a}\right)$. ...(1)

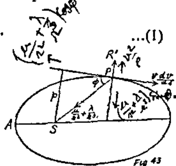


Fig 43

This V is the velocity of projection for our question under consideration.

Tangential equation of motion is

$$v\frac{dv}{ds}=-\left(\frac{\mu}{r^2}+\frac{\lambda}{r^3}\right)\cos\phi$$

$$=-\left(\frac{\mu}{r^2}+\frac{\lambda}{r^3}\right)\cdot\frac{dr}{ds}.$$

$$\therefore \int_V^v v\,dv=-\int_R^r\left(\frac{\mu}{r^2}+\frac{\lambda}{r^3}\right)dr.$$

$$\therefore \frac{v^2-V^2}{2}=-\left[-\frac{\mu}{r}-\frac{\lambda}{2r^2}\right]_R^r$$

or $$v^2-\mu\left(\frac{2}{R}-\frac{1}{a}\right)=2\left[\frac{\mu}{r}+\frac{\lambda}{2r^2}-\frac{\mu}{R}-\frac{\lambda}{2R^2}\right] \text{ by (1)}.$$

$$\therefore v^2=\frac{2\mu}{r}+\frac{\lambda}{r^2}-\frac{\lambda}{R^2}-\frac{\mu}{a}. \qquad ...(2)$$

If R' be the normal reaction then writing down the normal equation of motion, we get

$$\frac{v^2}{\rho}=\left(\frac{\mu}{r^2}+\frac{\lambda}{r^3}\right)\sin\phi-R'$$

$$\tfrac{1}{2}m(v^2-V^2)=-mgh=-mgr\cos\theta$$

or $$v^2=V^2-2gr\cos\theta. \quad \text{...(1)}$$

Now in order that the particle may make a complete revolution, the reaction at A the highest point should be+ive. Hence we shall calculate the reaction at A, where $\theta=0$.

Normal equation of motion is

$$m\frac{v^2}{\rho}=R-mg\sin(\theta-\phi)$$

or $$R=\frac{m}{\rho}(V^2-2gr\cos\theta)+mg\sin(\theta-\phi) \text{ by (1). ....(2)}$$

Now equation of the lemniscate is $r^2=a^2\cos 2\theta$.

$$\therefore\ 2\log r=\log a^2+\log\cos 2\theta.$$

$$\therefore\ 2.\frac{1}{r}.\frac{dr}{d\theta}=\frac{1}{\cos 2\theta}\times-2\sin 2\theta \text{ or } \cot\phi=-\tan 2\theta.$$

$$\therefore\ \phi=90+2\theta.$$

$$\therefore\ p=r\sin\phi=r\cos 2\theta=r.\frac{r^2}{a^2}=\frac{r^3}{a^2}.$$

$$\therefore\ \frac{dp}{dr}=\frac{3r^2}{a^2};\ \therefore\ \rho=r.\frac{dr}{dp}=\frac{r}{dp/dr}=r.\frac{a^2}{3r^2}=\frac{a^2}{3r}.$$

Now at the highest point $\theta=0$.

$$\therefore\ r=a \text{ and } \rho=\frac{a^2}{3r}=\frac{a^2}{3a}=\frac{a}{3}.$$

Also $$\phi=90+2\theta=90, \quad \because\ \theta=0.$$

Putting the above results in (2) we get

$$R=m.\frac{3}{a}(V^2-2ga.1)+mg\sin(0-90^\circ)$$

$$=m.\frac{3}{a}(V^2-2ga)-mg=\frac{m}{a}(3V^2-7ag).$$

Now R is+ive if $3V^2>7ag$. Hence proved.

Ex. 8. *A small bead moves on a thin elliptic wire under a force to the focus equal to* $\frac{\mu}{r^2}+\frac{\lambda}{r^3}$. *It is projected from a*

point on the wire distant R from the focus with the velocity which would cause it to describe the ellipse freely under a force $\frac{\mu}{r^2}$. *Show that the reaction of the wire is*

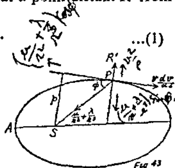
$$\frac{\lambda}{\rho}\left(\frac{1}{r^2}-\frac{1}{ar}+\frac{1}{R^2}\right)$$

where ρ *is the radius of curvature*

(Delhi Hons. 63; Agra 44, 57)

We know from last chapter that under inverse square law if the particle is to describe an elliptic orbit, then $v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right)$. If V be the velocity at a point distant R from the focus, then

$$V^2=\mu\left(\frac{2}{R}-\frac{1}{a}\right). \quad \ldots(1)$$



This V is the velocity of projection for our question under consideration.

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$$=-\left(\frac{\mu}{r^2}+\frac{\lambda}{r^3}\right)\cdot\frac{dr}{ds}.$$

$$\therefore \int_V^v v\,dv=-\int_R^r\left(\frac{\mu}{r^2}+\frac{\lambda}{r^3}\right)dr.$$

$$\therefore \frac{v^2-V^2}{2}=-\left[-\frac{\mu}{r}-\frac{\lambda}{2r^2}\right]_R^r$$

or

$$v^2-\mu\left(\frac{2}{R}-\frac{1}{a}\right)=2\left[\frac{\mu}{r}+\frac{\lambda}{2r^2}-\frac{\mu}{R}-\frac{\lambda}{2R^2}\right] \text{ by (1)}.$$

$$\therefore v^2=\frac{2\mu}{r}+\frac{\lambda}{r^2}-\frac{\lambda}{R^2}-\frac{\mu}{a}. \quad \ldots(2)$$

If R' be the normal reaction then writing down the normal equation of motion, we get

$$\frac{v^2}{\rho}=\left(\frac{\mu}{r^2}+\frac{\lambda}{r^3}\right)\sin\phi-R'$$

Again the normal equation of motion is

$$m\frac{v^2}{\rho}=R+mg\sin\phi+T\sin\phi.$$

Put for v^2, ρ, T and $\sin\phi$.

$$\therefore \quad R=m\cdot\frac{2gr}{a}(3a-r)\cdot\frac{1}{2r}\sqrt{\frac{a}{r}}-\left[mg+3mg\frac{(r-a)}{a}\right]\sqrt{\frac{a}{r}}$$

$$=\frac{mg}{a}\sqrt{\left(\frac{a}{r}\right)}\,[3a-r-a-2r+2a]$$

$$=\frac{mg}{\sqrt{(ar)}}\,[4a-3r].$$

Note. In case $3r > 4a$, then the reaction would change direction.

Note. We can deduce equation (2) from the equation of energy as follows :

$\frac{1}{2}m(v^2-V^2)=mg\,.\,AN$—work done by tension.

Now $V^2=4ag$ and $AN=AS+SN=a+r\cos\theta$.

But $\dfrac{2a}{r}=1-\cos\theta$ or $r-2a=r\cos\theta$; $\therefore$ $AN=r-a$.

Also work done by tension

$=$[mean of initial and final tensions]$\times$extension

$=\frac{1}{2}(T+T_0)(r-a)$,

where $T=\lambda\,.\,\dfrac{r-a}{a}=2mg.\dfrac{r-a}{a}$ and $T_0=0$ as the particle is projected from vertex A where $r=a$.

$\therefore$ work done by tension

$$=\tfrac{1}{2}\,.\,2mg\,\frac{r-a}{a}\,.\,(r-a)=\frac{mg}{a}(r-a)^2.$$

$$\therefore \quad \tfrac{1}{2}m(v^2-4ag)=mg\left[(r-a)-\frac{(r-a)^2}{a}\right].$$

$$\therefore \quad v^2=2ag+\frac{2g(r-a)(2a-r)}{a}$$

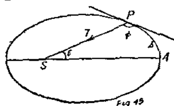
or $$v^2=\frac{2g}{a}[2a^2+3ar-2a^2-r^2]=\frac{2g}{a}[3ar-r^2].$$

Above is same as equation no. (2).

Ex. 10. *An elastic string of modulus λ is attached at one end to a focus of a smooth wire in the shape of an ellipse of latus rectum 2l and major axis 2a. The other end of the string is attached to a small ring of unit mass which can slide on the wire which is fixed* with its plane horizontal. *If the ring be slightly displaced from its position of unstable equilibrium at the end of major axis of the ellipse, show that its angular velocity about the focus when the string becomes slack is*

$$\sqrt{\left[\frac{\lambda l}{a^3}(a-l)\right]}.$$

The polar equation of the ellipse is $\frac{l}{r}=1-e\cos\theta$ referred to S as pole. The ring is of unit mass and the wire has its plane horizontal. The tangential equation of motion is



$$1 \cdot v \cdot \frac{dv}{ds}=-T\cos\phi$$

$$=-\frac{\lambda}{r_0}(r-r_0)\frac{dr}{ds},$$

where r_0 is the natural length of the string.

$$\therefore \quad \frac{v^2}{2}=-\frac{\lambda}{r_0}\left(\frac{r^2}{2}-r_0 r\right)+A.$$

At A, $\qquad r=a+ae$ and $v=0$.

$$\therefore \quad A=\frac{\lambda}{r_0}\left[\frac{(a+ae)^2}{2}-r_0(a+ae)\right].$$

$$\therefore \quad v^2=-\frac{\lambda}{r_0}(r^2-2rr_0)+\frac{\lambda}{r_0}[(a+ae)^2-2r_0(a+ae)] \quad ...(1)$$

Now radial velocity is $\frac{dr}{dt}$ and transverse velocity is $r\frac{d\theta}{dt}$.

$$\therefore \quad v^2=\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2.$$

Again differentiating, $\frac{l}{r}=1-e\cos\theta$ w.r.f. t, we get

$$-\frac{l}{r^2}\frac{dr}{dt}=e\sin\theta.\frac{d\theta}{dt} \quad \text{and} \quad \cos\theta=\frac{r-l}{er}.$$

$$\therefore \quad v^2=\frac{r^4}{l^2}e^2\sin^2\theta\left(\frac{d\theta}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2 \text{ by (2)}$$

$$=r^2\left(\frac{d\theta}{dt}\right)^2\left[\frac{r^2}{l^2}.e^2\left\{1-\frac{(r-l)^2}{e^2r^2}\right\}+1\right]$$

$$=\frac{r^2}{l^2}\left(\frac{d\theta}{dt}\right)^2[r^2e^2-(r-l)^2+l^2]$$

$$=\frac{r^2}{l^2}\left(\frac{d\theta}{dt}\right)^2[2rl-r^2(1-e^2)].$$

Again $$l=\frac{b^2}{a}=\frac{a^2(1-e^2)}{a}=a(1-e^2).$$

$$\therefore \quad v^2=\frac{r^2}{l^2}\left(\frac{d\theta}{dr}\right)^2\left(2rl-\frac{r^2.l}{a}\right)$$

$$=\frac{r^3}{al}\left(\frac{d\theta}{dt}\right)^2(2a-r). \quad \ldots(3)$$

Equating the values of v^2 from (1) and (3), we get

$$\frac{r^3}{al}\left(\frac{d\theta}{dt}\right)^2(2a-r)=\frac{\lambda}{r_0}[(a+ae)^2-2r_0(a+e)-r^2+2rr_0].$$

The string will be slack, when $r=r_0$ and hence the angular velocity $\frac{d\theta}{dt}$, when the string becomes slack, is obtained by putting $r=r_0$ in the above relation and we get

$$\frac{r_0^3}{al}\left(\frac{d\theta}{dt}\right)^2(2a-r_0)=\frac{\lambda}{r_0}[(a+ae)(a+ae-2r_0)-r_0^2+2r_0^2]$$

$$=\frac{\lambda}{r_0}[(a+ae)(a+ae-2r_0)+r_0^2].$$

We are not given the value of r_0, the natural length of the string. Let us choose it to be a to get the required answer.

$$\therefore \quad \frac{a^3}{at}\left(\frac{d\theta}{dt}\right)^2(2a-a)=\frac{\lambda}{a}\left[(a+ae)(ae-a)+a^2\right]$$

or
$$\frac{a^3}{l}\left(\frac{d\theta}{dt}\right)^2=\frac{\lambda}{a}(a^2e^2-a^2+a^2)=\lambda ae^2.$$

$$\therefore \quad \frac{d\theta}{dt}=\sqrt{\left(\frac{l\lambda}{a^3}\cdot ae^2\right)}.$$

But
$$l=\frac{b^2}{a}=\frac{a^2(1-e^2)}{a}=a-ae^2.$$

$$\therefore \quad ae^2=a-l.$$

$$\therefore \quad \frac{d\theta}{dt}=\sqrt{\left[\frac{l\lambda}{a^3}(a-l)\right]}. \qquad \text{Proved.}$$

Ex. 11. *A small heavy bead is threaded on a rough circular wire which is clamped at the ends of a diameter, so that its plane is horizontal. The bead is projected with velocity V from one end of the clamps and just comes to rest at the other. Prove that the velocity of projection is $\sqrt{(ga \sinh 2\pi\mu)}$, where a is the radius of the wire.*

A bead threaded on the wire means that the bead cannot leave the wire. The bead presses the wire outwardly as well as vertically downwards. Hence there will be two reactions in this case : one normal reaction R towards the centre and the other vertical equal to mg. Hence the total reaction on the bead is $\sqrt{(R^2+m^2g^2)}$. Therefore the force of friction is

$$\mu\sqrt{(R^2+m^2g^2)}.$$

Equations of motion are

$$mv\frac{dv}{ds}=-\mu\sqrt{(R^2+m^2g^2)} \qquad \ldots(1)$$

and
$$m\frac{v^2}{\rho}=R. \qquad \ldots(2)$$

Eliminating R between (1) and (2) and putting $\rho=a$ for a circle,

$$mv\frac{dv}{ds}=-\mu\sqrt{\left(m^2\frac{v^4}{a^2}+m^2g^2\right)}=-\mu\frac{m}{a}\sqrt{(v^4+g^2a^2)}$$

or $$\frac{1}{2}\frac{dv^2}{ds}=-\frac{\mu}{a}\sqrt{(v^4+g^2a^2)}$$

or $$\frac{dv^2}{d\theta}\cdot\frac{d\theta}{ds}=-\frac{2\mu}{a}\sqrt{(v^4+g^2a^2)}.$$

In a circle $s=a\theta$, $\therefore$ $\frac{ds}{d\theta}=a$ or $\frac{d\theta}{ds}=\frac{1}{a}$.

$$\therefore\quad\frac{dv^2}{d\theta}=-2\mu\sqrt{(v^4+g^2a^2)}.$$

Putting $v^2=z$, we get

$$\frac{dz}{d\theta}=-2\mu\sqrt{(z^2+g^2a^2)}$$

or $$\frac{dz}{\sqrt{(z^2+g^2a^2)}}=-2\mu\ d\theta.$$

Integrating, we get $\sinh^{-1}\frac{z}{ga}=-2\mu\theta+A$.

We are given that it comes to rest at the other end of the diameter, *i.e.* when $\theta=\pi$, $z=v^2=0$ and $\sinh^{-1}0=0$.

$$\therefore\quad 0=-2\mu\pi+A\quad\text{or}\quad A=2\mu\pi.$$

$$\therefore\quad\frac{z}{ga}=\sinh(2\mu\pi-2\mu\theta).$$

Initially when $\theta=0$, $z=v^2=V^2$ given

$$V^2=ga\sinh(2\mu\pi)$$

or $$V=\sqrt{\{ga\sinh(2\mu\pi)\}}.$$

Ex. 12. *A particle falls from a position of limiting equilibrium near the top of a nearly smooth glass sphere. Show that it will leave the sphere at the point whose radius is inclined to the vertical at an angle* $\alpha+\mu\left[2-\frac{4\alpha}{3\sin\alpha}\right]$ *where* $\cos\alpha=\frac{2}{3}$ *and* μ *is the small coefficient of friction.*

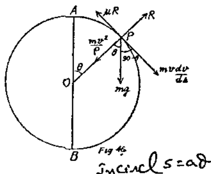
The equations of motion are

$$m\frac{dv}{ds}=-\mu R+mg\sin\theta, \quad ..(1)$$

$$m\frac{v^2}{\rho}=mg\cos\theta-R. \quad ...(2)$$

Eliminating R, we get

$$mv\frac{dv}{ds}=\mu m\frac{v^2}{\rho}-\mu mg\cos\theta$$
$$+mg\sin\theta$$



Incircle $s=a\theta$ $\quad \frac{ds}{d\theta}=a$

or $$v\frac{dv}{d\theta}\cdot\frac{d\theta}{ds}-\mu\frac{v^2}{a}=g(\sin\theta-\mu\cos\theta)$$

or $$\frac{dv^2}{d\theta}-2\mu v^2=2ag(\sin\theta-\mu\cos\theta) \quad \because \frac{ds}{d\theta}=a.$$

Putting $v^2=z$, we get

$$\frac{dz}{d\theta}-2\mu z=2ag(\sin\theta-\mu\cos\theta).$$

Above is linear differential equation and

$$\text{I. F.}=e^{\int P\,d\theta}=e^{\int -2\mu\,d\theta}=e^{-2\mu\theta}.$$

Multiplying both sides by I. F. and integrating, we get

$$z\,.\,e^{-2\mu\theta}=2ag\int e^{-2\mu\theta}(\sin\theta-\mu\cos\theta)\,d\theta. \quad ...(3)$$

Now $\displaystyle\int e^{(a+ib)x}\,dx=\frac{e^{(a+ib)x}}{a+ib}$

or $\displaystyle\int e^{ax}(\cos bx+i\sin bx)\,dx=\frac{e^{ax}(\cos bx+i\sin bx)}{(a^2+b^2)}\,.(a-ib).$

Equating real and imaginary parts, we get

$$\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{(a^2+b^2)}(a\sin bx-b\cos bx)$$
$$=\frac{e^{ax}}{r}\sin(bx-\alpha),$$

$$\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{(a^2+b^2)}(a\cos bx+b\sin bx)$$

$$=\frac{e^{ax}}{r}\cos(bx-\alpha),$$

where $\tan\alpha=\frac{b}{a}$ and $r=\sqrt{(a^2+b^2)}$.

Applying the above integrals, we get from (3)

$$v^2e^{-2\mu\theta}=2ag\left[e^{-2\mu\theta}\left(\frac{-2\mu\sin\theta-\cos\theta}{1+4\mu^2}\right)\right.$$
$$\left.-\mu\,.\,e^{-2\mu\theta}\left(\frac{-2\mu\cos\theta+\sin\theta}{1+4\mu^2}\right)+A\right].$$

$$\therefore\quad v^2e^{-2\mu\theta}=\frac{2ag\,.\,e^{-2\mu\theta}}{(1+4\mu^2)}\,[-3\mu\sin\theta-(1-2\mu^2)\cos\theta]+A.$$

Since μ is small, neglecting μ^2 and higher powers, we have

$$e^{-2\mu\theta}=1-2\mu\theta+\frac{(2\mu\theta)^2}{2\,!}-\ldots=1-2\mu\theta,$$

$$(1+4\mu^2)^{-1}=1-4\mu^2+\ldots=1.$$

$$\therefore\quad v^2(1-2\mu\theta)=2ag\,(1-2\mu\theta)\,[-3\mu\sin\theta-\cos\theta]+A$$

or $$v^2(1-2\mu\theta)=2ag\,[-3\mu\sin\theta-(1-2\mu\theta)\cos\theta]+A\ldots(4)$$

Initially when $\theta=0$ at the top, $v=0$.

$$\therefore\quad 0=2ag\,[0-1]+A\quad\text{or}\quad A=2ag.$$

Putting for A in (4), we get

$$v^2=2ag\,[-3\mu\sin\theta-(1-2\mu\theta)\cos\theta+1]\,(1-2\mu\theta)^{-1}$$
$$=2ag\,[-3\mu\sin\theta-(1-2\mu\theta)\cos\theta+1]\,(1+2\mu\theta)$$
$$=2ag\,[-3\mu\sin\theta-1\,.\cos\theta+(1+2\mu\theta)]\,.\text{neglect }\mu^2\text{ etc.}$$

Now from (2),

$$R=mg\cos\theta-\frac{mv^2}{a}$$

or $$R=mg\cos\theta-\frac{m}{a}.2ag\,[-3\mu\sin\theta-\cos\theta+(1+2\mu\theta)]$$
$$=mg\,[\cos\theta+6\mu\sin\theta+2\cos\theta-2\,(1+2\mu\theta)].$$

The particle will leave the sphere when $R=0$.

$$\therefore\quad 3\cos\theta+6\mu\sin\theta=2\,(1+2\mu\theta).\qquad\ldots(5)$$

To a first approximation, taking $\mu=0$,
$3\cos\theta=2$ or $\cos\theta=\frac{2}{3}=\cos\alpha$ (giv

Hence we can choose $\theta=\alpha+\beta$ where
better approximation.

Putting for θ in (5), we get

$$3\cos(\alpha+\beta)+6\mu\sin(\alpha+\beta)=2+4\mu(\alpha+\beta)$$

or $3(\cos\alpha.1-\beta\sin\alpha)+6\mu(\sin\alpha.1+\beta\cos\alpha)=2+4\mu\alpha+4\mu\beta$.

Since β is small, we have taken $\cos\beta=1$ and $\sin\beta=\beta$.

$\therefore\ \beta(6\mu\cos\alpha-3\sin\alpha-4\mu)=2+4\mu\alpha-6\mu\sin\alpha-3\cos\alpha$.

Again β is small; therefore neglecting $\mu.\beta$, which is a small quantity of 2nd order, we get

$$\beta(-3\sin\alpha)=2+4\mu\alpha-6\mu\sin\alpha-3.\tfrac{2}{3},\qquad \because\ \cos\alpha=\tfrac{2}{3}.$$

$$\therefore\quad \beta=\frac{6\mu\sin\alpha-4\mu\alpha}{3\sin\alpha}=\mu\left(2-\frac{4}{3}\ \frac{\alpha}{\sin\alpha}\right).$$

$$\therefore\quad \theta=\alpha+\beta=\alpha+\mu\left(2-\frac{4}{3}.\frac{\alpha}{\sin\alpha}\right).$$

T 1970

Motion on a Circle

Ex. 13. *A particle is projected horizontally with velocity V along the inside of a rough vertical circle from the lowest point. Prove that if it completes the circuit, it will return to the lowest point with a velocity v given by*

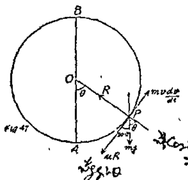
$$v^2=V^2e^{-4\pi\mu}+\frac{2ga}{1+4\mu^2}(1-2\mu^2)(1-e^{-4\pi\mu}).$$

(Delhi Hons. 53)

Equations of motion are

$$mv\frac{dv}{ds}=-\mu R-mg\sin\theta, \quad ...(1)$$

$$m\frac{v^2}{\rho}=R-mg\cos\theta. \quad ...(2)$$



Eliminating R between the above, we get

$$\frac{m}{2}\frac{dv^2}{d\theta}\cdot\frac{d\theta}{ds}=-\mu m\frac{v^2}{a}-\mu mg\cos\theta$$
$$-mg\sin\theta$$

or $$\frac{dv^2}{d\theta}+2\mu v^2=-2ga\,(\mu\cos\theta+\sin\theta)\,;\quad \because\ \frac{d\theta}{ds}=\frac{1}{a}.$$

Above is linear and I. F. $=e^{\int 2\mu\,d\theta}=e^{2\mu\theta}$.

Multiplying both sides by $e^{2\mu\theta}$ *and integrating, we get*

$$v^2.e^{2\mu\theta}=-2ga\int[\mu e^{2\mu\theta}\cos\theta+e^{2\mu\theta}\sin\theta]\,d\theta$$

or $$v^2.e^{2\mu\theta}=-2ga\left[\mu\,\frac{e^{2\mu\theta}}{1+4\mu^2}(2\mu\cos\theta+\sin\theta)+\frac{e^{2\mu\theta}}{1+4\mu^2}(2\mu\sin\theta-\cos\theta)\right]+C$$

or $$v^2.e^{2\mu\theta}=-2ga\,\frac{e^{2\mu\theta}}{1+4\mu^2}[3\mu\sin\theta-(1-2\mu^2)\cos\theta]+C. \quad ...(3)$$

Initially when $\theta=0$, $v=V$ given.

$$\therefore\quad V^2.1=-\frac{2ga}{1+4\mu^2}[0-(1-2\mu^2)]+C.$$

$$\therefore\quad C=V^2-\frac{2ga}{1+4\mu^2}(1-2\mu^2).$$

$$\therefore\quad v^2e^{2\mu\theta}=-2ga\,\frac{e^{2\mu\theta}}{1+4\mu^2}[3\mu\sin\theta-(1-2\mu^2)\cos\theta]+V^2-\frac{2ga}{1+4\mu^2}(1-2\mu^2).$$

When the particle returns to the lowest point, it would have described an angle $\theta=2\pi$ and let the velocity be v. Then putting $\theta=2\pi$, we get

$$v^2e^{4\pi\mu}=\frac{-2ga}{1+4\mu^2}\,e^{4\pi\mu}[0-(1-2\mu^2).1]+V^2-\frac{2ga}{1+4\mu^2}(1-2\mu^2).$$

Dividing throughout by $e^{4\pi\mu}$, we get

$$v^2=V^2e^{-4\pi\mu}+\frac{2ga}{1+4\mu^2}(1-2\mu^2)[1-e^{-4\pi\mu}].$$

Ex. 14. *A ring which can slide on a rough circular wire in a vertical plane is projected from the lowest point with such*

a velocity as will take it to the horizontal diameter. If the ring returns to the lowest point, show that its velocity on arrival is to its velocity of projection as

$$[1-2\mu^2-3\mu e^{-\mu\pi}]^{1/2} : [1-2\mu^2+3\mu e^{\mu\pi}]^{1/2}. \quad \text{(Agra 53)}$$

Proceeding exactly as in last question, we have the same equations (1) and (2) when the particle is projected from the lowest point. Hence we have the equation (3) after integration as under :—

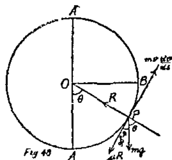


Fig 48

$$v^2 e^{2\mu\theta} = -2ga \cdot \frac{e^{2\mu\theta}}{1+4\mu^2} [3\mu \sin\theta - (1-2\mu^2)\cos\theta)] + C.$$

Since particle reaches only upto the end of horizontal diameter at B *i.e.* $v=0$ when $\theta=\pi/2$,

$$\therefore \quad 0 = -2ga \cdot \frac{e^{\mu\pi}}{1+4\mu^2} \cdot 3\mu + C. \quad \therefore \quad C = 2ga \frac{e^{\mu\pi}}{1+4\mu^2} 3\mu.$$

$$\therefore \quad v^2 e^{2\mu\theta} = -2ga \frac{e^{2\mu\theta}}{1+4\mu^2} [3\mu \sin\theta - (1-2\mu^2)\cos\theta] + 2ga \cdot \frac{e^{\mu\pi}}{1+4\mu^2} 3\mu. \quad \ldots(1)$$

If V be the velocity of projection at A, then putting $\theta=0$ and $v=V$, we get

$$V^2 = \frac{2ga}{1+4\mu^2} [1-2\mu^2+3\mu e^{\mu\pi}]. \quad \ldots(2)$$

2nd Case. Now the particle retraces its path from B where $\theta=\pi/2$ and velocity is zero (same as in case 1). Considering the position P where $\angle AOP=\theta$ we will have same forces except that the direction of the force of friction will be opposite to that what it was in the first case. Hence in order to obtain the velocity at any point P, we have to simply

or $\dfrac{dv^2}{d\theta}+2\mu v^2=-2ga\,(\mu\cos\theta+\sin\theta)$; $\because\ \dfrac{d\theta}{ds}=\dfrac{1}{a}$.

Above is linear and I. F. $=e^{\int 2\mu\, d\theta}=e^{2\mu\theta}$.

Multiplying both sides by $e^{2\mu\theta}$ and integrating, we get

$$v^2.e^{2\mu\theta}=-2ga\int[\mu e^{2\mu\theta}\cos\theta+e^{2\mu\theta}\sin\theta]\,d\theta$$

or $$v^2.e^{2\mu\theta}=-2ga\left[\mu\,\frac{e^{2\mu\theta}}{1+4\mu^2}(2\mu\cos\theta+\sin\theta)+\frac{e^{2\mu\theta}}{1+4\mu^2}(2\mu\sin\theta-\cos\theta)\right]+C$$

or $$v^2.e^{2\mu\theta}=-2ga\,\frac{e^{2\mu\theta}}{1+4\mu^2}[3\mu\sin\theta-(1-2\mu^2)\cos\theta]+C. \quad \ldots(3)$$

Initially when $\theta=0$, $v=V$ given.

$$\therefore\quad V^2.1=-\frac{2ga}{1+4\mu^2}[0-(1-2\mu^2)]+C.$$

$$\therefore\quad C=V^2-\frac{2ga}{1+4\mu^2}(1-2\mu^2).$$

$$\therefore\quad v^2e^{2\mu\theta}=-2ga\,\frac{e^{2\mu\theta}}{1+4\mu^2}[3\mu\sin\theta-(1-2\mu^2)\cos\theta]+V^2-\frac{2ga}{1+4\mu^2}(1-2\mu^2).$$

When the particle returns to the lowest point, it would have described an angle $\theta=2\pi$ and let the velocity be v. Then putting $\theta=2\pi$, we get

$$v^2e^{4\pi\mu}=\frac{-2ga}{1+4\mu^2}e^{4\pi\mu}[0-(1-2\mu^2).1]+V^2-\frac{2ga}{1+4\mu^2}(1-2\mu^2).$$

Dividing throughout by $e^{4\pi\mu}$, we get

$$v^2=V^2e^{-4\pi\mu}+\frac{2ga}{1+4\mu^2}(1-2\mu^2)[1-e^{-4\pi\mu}].$$

Ex. 14. *A ring which can slide on a rough circular wire in a vertical plane is projected from the lowest* [illegible] *such*

a velocity as will take it to the horizontal diameter. If the ring returns to the lowest point, show that its velocity on arrival is to its velocity of projection as

$$[1-2\mu^2-3\mu e^{-\mu\pi}]^{1/2} : [1-2\mu^2+3\mu e^{\mu\pi}]^{1/2} \qquad \text{(Agra 53)}$$

Proceeding exactly as in last question, we have the same equations (1) and (2) when the particle is projected from the lowest point. Hence we have the equation (3) after integration as under :—

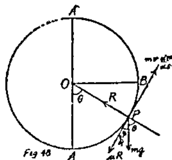


Fig 48

$$v^2 e^{2\mu\theta} = -2ga \cdot \frac{e^{2\mu\theta}}{1+4\mu^2} [3\mu \sin\theta - (1-2\mu^2)\cos\theta)] + C.$$

Since particle reaches only upto the end of horizontal diameter at B *i.e.* $v=0$ when $\theta=\pi/2$,

$$\therefore \quad 0 = -2ga \cdot \frac{e^{\mu\pi}}{1+4\mu^2} \cdot 3\mu + C. \quad \therefore \quad C = 2ga \frac{e^{\mu\pi}}{1+4\mu^2} 3\mu.$$

$$\therefore \quad v^2 e^{2\mu\theta} = -2ga \frac{e^{2\mu\theta}}{1+4\mu^2} [3\mu \sin\theta - (1-2\mu^2)\cos\theta]$$

$$+2ga \cdot \frac{e^{\mu\pi}}{1+4\mu^2} 3\mu. \qquad \ldots(1)$$

If V be the velocity of projection at A, then putting $\theta=0$ and $v=V$, we get

$$V^2 = \frac{2ga}{1+4\mu^2} [1-2\mu^2+3\mu e^{\mu\pi}]. \qquad \ldots(2)$$

2nd Case. Now the particle retraces its path from B where $\theta=\pi/2$ and velocity is zero (same as in case 1). Considering the position P where $\angle AOP=\theta$ we will have same forces except that the direction of the force of friction will be opposite to that what it was in the first case. Hence in order to obtain the velocity at any point P, we have to simply

or $$\frac{dv^2}{d\theta}+2\mu v^2=-2ga\ (\mu\cos\theta+\sin\theta)\ ;\quad \because\ \frac{d\theta}{ds}=\frac{1}{a}.$$

Above is linear and I. F. $=e^{\int 2\mu\, d\theta}=e^{2\mu\theta}$.

Multiplying both sides by $e^{2\mu\theta}$ and integrating, we get

$$v^2 . e^{2\mu\theta}=-2ga \int [\mu e^{2\mu\theta}\cos\theta+e^{2\mu\theta}\sin\theta]\, d\theta$$

or $$v^2 . e^{2\mu\theta}=-2ga\left[\mu\,\frac{e^{2\mu\theta}}{1+4\mu^2}(2\mu\cos\theta+\sin\theta)+\frac{e^{2\mu\theta}}{1+4\mu^2}(2\mu\sin\theta-\cos\theta)\right]+C$$

or $$v^2 . e^{2\mu\theta}=-2ga\,\frac{e^{2\mu\theta}}{1+4\mu^2}[3\mu\sin\theta-(1-2\mu^2)\cos\theta]+C. \qquad \ldots(3)$$

Initially when $\theta=0$, $v=V$ given.

$$\therefore\ V^2 . 1=-\frac{2ga}{1+4\mu^2}[0-(1-2\mu^2)]+C.$$

$$\therefore\ C=V^2-\frac{2ga}{1+4\mu^2}(1-2\mu^2).$$

$$\therefore\ v^2 e^{2\mu\theta}=-2ga\,\frac{e^{2\mu\theta}}{1+4\mu^2}[3\mu\sin\theta-(1-2\mu^2)\cos\theta]+V^2-\frac{2ga}{1+4\mu^2}(1-2\mu^2).$$

When the particle returns to the lowest point, it would have described an angle $\theta=2\pi$ and let the velocity be v. Then putting $\theta=2\pi$, we get

$$v^2 e^{4\pi\mu}=\frac{-2ga}{1+4\mu^2}e^{4\pi\mu}\{0-(1-2\mu^2).1\}+V^2-\frac{2ga}{1+4\mu^2}(1-2\mu^2).$$

Dividing throughout by $e^{4\pi\mu}$, we get

$$v^2=V^2 e^{-4\pi\mu}+\frac{2ga}{1+4\mu^2}(1-2\mu^2)[1-e^{-4\pi\mu}].$$

Ex. 14. *A ring which can slide on a rough circular wire in a vertical plane is projected from the lowest point with such*

a velocity as will take it to the horizontal diameter. If the ring returns to the lowest point, show that its velocity on arrival is to its velocity of projection as

$$[1-2\mu^2-3\mu e^{-\mu\pi}]^{1/2} : [1-2\mu^2+3\mu e^{\mu\pi}]^{1/2}. \quad \text{(Agra 53)}$$

Proceeding exactly as in last question, we have the same equations (1) and (2) when the particle is projected from the lowest point Hence we have the equation (3) after integration as under :—

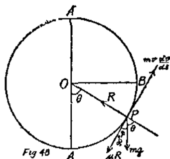


Fig 48

$$v^2e^{2\mu\theta}=-2ga.\frac{e^{2\mu\theta}}{1+4\mu^2}[3\mu\sin\theta-(1-2\mu^2)\cos\theta)]+C$$

Since particle reaches only upto the end of horizontal diameter at B *i.e.* $v=0$ when $\theta=\pi/2$,

$$\therefore \quad 0=-2ga.\frac{e^{\mu\pi}}{1+4\mu^2}.3\mu+C. \quad \therefore \quad C=2ga\frac{e^{\mu\pi}}{1+4\mu^2}3\mu.$$

$$\therefore \quad v^2e^{2\mu\theta}=-2ga\frac{e^{2\mu\theta}}{1+4\mu^2}[3\mu\sin\theta-(1-2\mu^2)\cos\theta]$$

$$+2ga.\frac{e^{\mu\pi}}{1+4\mu^2}3\mu. \quad \ldots(1)$$

If V be the velocity of projection at A, then putting $\theta=0$ and $v=V$, we get

$$V^2=\frac{2ga}{1+4\mu^2}[1-2\mu^2+3\mu e^{\mu\pi}]. \quad \ldots(2)$$

2nd Case. Now the particle retraces its path from B where $\theta=\pi/2$ and velocity is zero (same as in case 1). Considering the position P where $\angle AOP=\theta$ we will have same forces except that the direction of the force of friction will be opposite to that what it was in the first case. Hence in order to obtain the velocity at any point P, we have to simply

change the sign of μ in (1). If V_1 be its velocity when it returns to A where $\theta=0$, then writing $-\mu$ for μ in (2), we get

$$V_1^2=\frac{2ag}{1+4\mu^2}\,(1-2\mu^2-3\mu e^{-\mu\pi}]. \qquad \ldots(3)$$

Hence from (2) and (3), we get

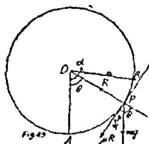
$$V:V_1::[1-2\mu^2+3\mu e^{\mu\pi}]^{1/2}:[1-2\mu^2-3\mu e^{-\mu\pi}]^{1/2}$$

or $\quad V_1\cdot V$ as given. *Proved.*

Note. See also question 16 where V_1^2 is found independently.

Ex. 15. *A particle is projected horizontally from the lowest point of a rough sphere of radius a. After describing an arc less than a quadrant, it returns and comes to rest at the lowest point. Show that the initial velocity must be* $\sin\alpha\sqrt{\left(2ga\cdot\frac{1+\mu^2}{1-2\mu^2}\right)}$ *where μ is the coefficient of friction and α is the arc through which the particle moves* **(Agra 58)**

Since $s=a\theta$, hence the particle comes to rest when $\theta=\alpha$ and retraces its path. Arguing exactly as in Q. 13 and writing similar equations and integrating, we have equation (3) of Q. 13 as under.



$$v^2.e^{2\mu\theta}=-2ga\,\frac{e^{2\mu\theta}}{1+4\mu^2}$$
$$\times[3\mu\sin\theta-(1-2\mu^2)\cos\theta]+C.$$

When $\theta=\alpha$, $v=0$; $\therefore$ $C=$etc.

$$\therefore\quad v^2e^{2\mu\theta}=-2ga\,\frac{e^{2\mu\theta}}{1+4\mu^2}$$
$$\times[3\mu\sin\theta-(1-2\mu^2)\cos\theta]$$
$$+2ga\,\frac{e^{2\mu\alpha}}{1+4\mu^2}\,[3\mu\sin\alpha$$
$$-(1-2\mu^2)\cos\alpha]. \qquad \ldots(1)$$

Above gives us the velocity at any point in upward

motion. If V be the velocity of projection, $v=V$ when $\theta=0$.

$$\therefore \quad V^2=\frac{2ga}{1+4\mu^2}(1-2\mu^2)+\frac{2ga}{1+4\mu^2}.e^{2\mu\alpha}$$
$$\times[3\mu \sin \alpha-(1-2\mu^2)\cos \alpha]. \quad \ldots(2)$$

2nd Case. When the particle retraces its path from $\theta=\alpha$ where $v=0$, then putting $\mu=-\mu$ in (1), we get the velocity at any point P where $\angle AOP=\theta$ as under.

$$v^2e^{-2\mu\theta}=-2ga\frac{e^{-2\mu\theta}}{1+4\mu^2}[-3\mu \sin \theta-(1-2\mu^2)\cos \theta]$$
$$+2ga\frac{e^{-2\mu\alpha}}{1+4\mu^2}[-3\mu \sin \alpha-(1-2\mu^2)\cos \alpha].$$

Since the particle comes to rest at the lowest point
$\therefore \quad v=0$ when $\theta=0$.

$$\therefore \quad 0=-\frac{2ga}{1+4\mu^2}[0-(1-2\mu^2)]+2ga\frac{e^{-2\mu\alpha}}{1+4\mu^2}$$
$$\times[-3\mu \sin \alpha-(1-2\mu^2)\cos \alpha]$$

or $$\frac{1-2\mu^2}{1+4\mu^2}=e^{-2\mu\alpha}[3\mu \sin \alpha+(1-2\mu^2)\cos \alpha]$$

or $$\frac{e^{2\mu\alpha}}{1+4\mu^2}=\frac{1}{1-2\mu^2}[3\mu \sin \alpha+(1-2\mu^2)\cos \alpha]. \quad \ldots(3)$$

Putting for $\frac{e^{2\mu\alpha}}{1+4\mu^2}$ from (3) in (2), we get

$$V^2=\frac{2ga}{1+4\mu^2}(1-2\mu^2)+\frac{2ga}{1-2\mu^2}[9\mu^2 \sin^2 \alpha-(1-2\mu^2)^2\cos^2 \alpha]$$
$$=\frac{2ga}{(1+4\mu^2)(1-2\mu^2)}[(1-2\mu^2)^2(1-\cos^2 \alpha)+9\mu^2 \sin^2 \alpha]$$

or $$V^2=\frac{2ga}{(1+4\mu^2)(1-2\mu^2)}\sin^2 \alpha[1-4\mu^2+4\mu^4+9\mu^2]$$

or $$V^2=\frac{2ga \sin^2 \alpha}{(1+4\mu^2)(1-2\mu^2)}(1+\mu^2)(1+4\mu^2)$$

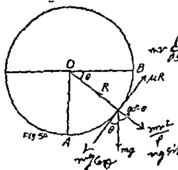
or $$V^2=2ga \sin^2 \alpha\frac{1+\mu^2}{1-2\mu^2}.$$

$$\therefore \quad V=\sin\alpha\sqrt{\left(2ga\,\frac{1+\mu^2}{1-2\mu^2}\right)}.$$

Ex. 16. *A bead slides down a rough circular wire, which is in a vertical plane starting from rest at the end of the horizontal diameter. When it has described an angle θ about the centre, show that the square of its angular velocity is*

$$\frac{2g}{a(1+4\mu^2)}\left[(1-2\mu^2)\sin\theta+3\mu\,(\cos\theta-e^{-2\mu\theta})\right]$$

where μ is the coefficient of friction and a the radius of the circle. **(Cal. Hons. 57 ; Agra 49, 52, 53, 55, 61, 65)**



As usual we have the following equations of motion

$$mv\,\frac{dv}{ds}=mg\cos\theta-\mu R, \quad \ldots(1)$$

$$m\,\frac{v^2}{\rho}=R-mg\sin\theta. \quad \ldots(2)$$

Eliminating R, we have

$$\frac{m}{2}\frac{dv^2}{d\theta}\cdot\frac{d\theta}{ds}=mg\cos\theta-\mu m\,\frac{v^2}{a}-\mu mg\sin\theta$$

or $$\frac{1}{2a}\,\frac{dv^2}{d\theta}+\mu\,\frac{v^2}{a}=g\,(\cos\theta-\mu\sin\theta)$$

or $$\frac{dv^2}{d\theta}+2\mu v^2=2ag\,(\cos\theta-\mu\sin\theta).$$

Above is linear differential equation and I. F. is $e^{2\mu\theta}$.

Multiplying both sides by $e^{2\mu\theta}$ and integrating, we get

$$v^2 . e^{2\mu\theta}=2ga\int\{e^{2\mu\theta}\cos\theta-\mu e^{2\mu\theta}\sin\theta\}\,d\theta$$

or $$v^2 . e^{2\mu\theta}=2ga\,.\,\frac{e^{2\mu\theta}}{1+4\mu^2}\left[(2\mu\cos\theta+\sin\theta)-\mu\,(2\mu\sin\theta-\cos\theta)\right]+C$$

or $v^2 . e^{2\mu\theta} = 2ga . \frac{e^{2\mu\theta}}{1+4\mu^2} [3\mu \cos\theta + (1-2\mu^2) \sin\theta] + C.$

When $\theta=0$, $v=0$.

$$\therefore \quad 0 = \frac{2ga}{1+4\mu^2} . 3\mu + C ; \quad \therefore \quad C = -\frac{2ga}{1+4\mu^2} . 3\mu.$$

$$\therefore \; v^2 . e^{2\mu\theta} = \frac{2ga}{1+4\mu^2} e^{2\mu\theta} [3\mu \cos\theta + (1-2\mu^2) \sin\theta] - \frac{2ga}{1+4\mu^2} . 3\mu$$

or $$v^2 = \frac{2ga}{1+4\mu^2} [3\mu (\cos\theta - e^{-2\mu\theta}) + (1-2\mu^2) \sin\theta] \quad \ldots (3)$$

But in a circle, $s = a\theta$; $v = \frac{ds}{dt} = a \frac{d\theta}{dt}$.

Putting $v = a \left(\frac{d\theta}{dt}\right)$ in (3), we get the value of $\left(\frac{d\theta}{dt}\right)^2$ as given.

Note. In order to find the velocity V_1 at the lowest point, we put $\theta = \frac{\pi}{2}$ in (3).

$\therefore \quad V_1^2 = \frac{2ga}{1+4\mu^2} [1 - 2\mu^2 - 3\mu e^{-\mu\pi}]$ which is same as V_1^2 found in Ex. 14 result (3), P. 216.

Ex. 17. *A particle is projected along the inner surface of a rough sphere and is acted on by no forces. Show that it will return to the point of projection at the end of time $\frac{a}{\mu V} (e^{2\mu\pi} - 1)$, where a is the radius of the sphere, V is the velocity of projection and μ is the coefficient of friction.*

(Sagar 62, Nagpur 58)

Since the particle is not acted on by other forces, we have the following equations of motion :—

$$mv \frac{dv}{ds} = -\mu R \text{ and } m \frac{v^2}{\rho} = R.$$

$$\frac{d\theta}{ds}=-\mu \,.\, m\,\frac{v^2}{a}\;;\quad \because\quad \rho=a.$$

$$\frac{\mu}{a}v^2=0 \text{ or } \frac{dv^2}{d\theta}+2\mu v^2=0.$$

$\therefore$ I. F. is $e^{2\mu\theta}$.

...plying both sides by $e^{2\mu\theta}$ and integrating, we get

$$v^2e^{2\mu\theta}=C.$$

Initially when $\theta=0$, then $v=V$. $\therefore\quad C=V^2$.

$$\therefore\quad v^2e^{2\mu\theta}=V^2 \quad\text{or}\quad v\,.\,e^{\mu\theta}=V.$$

But $\qquad v=\dfrac{ds}{dt}=a\,\dfrac{d\theta}{dt}.$

$$\therefore\quad a\frac{d\theta}{dt}\,.\,e^{\mu\theta}=V\;;\quad\therefore\quad a\int_0^{2\pi}e^{\mu\theta}\,d\theta=\int_0^T V\,dt,$$

where T is the time when the particle returns to the point of projection after describing an angle 2π

$$\therefore\quad a\left[\frac{e^{\mu\theta}}{\mu}\right]_0^{2\pi}=VT\;;\quad\therefore\quad T=\frac{a}{\mu V}\,[e^{2\mu\pi}-1].$$

Ex. 18. *A particle under no forces is projected with velocity V in a rough tube in the form of an equiangular spiral at a distance a from the pole and towards the pole. Show that it will arrive at the pole in time.*

$$\frac{a}{V}\cdot\frac{1}{\cos\alpha-\mu\cos\alpha}$$

α being the angle of the spiral and ($\mu<\cot\alpha$) the coefficient of friction. (Agra 63)

The equation of the equiangular spiral is $r=ae^{\theta\cot\alpha}$.

Also $\phi=\alpha$ and $\psi=\theta+\phi=\theta+\alpha$ and its pedal equation is $p=r\sin\alpha$; $\therefore\ \dfrac{dp}{dr}=\sin\alpha$ and $\rho=r\dfrac{dr}{dp}=r\operatorname{cosec}\alpha$.

The particle is moving under no forces and hence we have the following equations of motion

$$mv\frac{dv}{ds}=\mu R \quad\text{and}\quad m\frac{v^2}{\rho}=R$$

or $$mv\frac{dv}{ds}=\mu . m\frac{v^2}{\rho}$$

or $$\frac{1}{2}\frac{dv^2}{d\theta}\cdot\frac{d\theta}{ds}=\frac{\mu v^2}{r \operatorname{cosec} \alpha}$$

or $$\frac{1}{2}\frac{dv^2}{d\theta}\cdot r\frac{d\theta}{ds}=\frac{\mu v^2}{\operatorname{cosec} \alpha}.$$

But $\tan\phi=r\frac{d\theta}{dr}$; $\therefore$ $\sin\phi=r\frac{d\theta}{ds}=\sin\alpha$; $\because$ $\phi=\alpha$.

$$\therefore \quad \frac{1}{2}\frac{dv^2}{d\theta}.\sin\alpha-\mu v^2.\sin\alpha=0$$

or $$\frac{dv^2}{d\theta}-2\mu v^2=0.$$

Above is linear and I. F. $=e^{-2\mu\theta}$.

Multiplying both sides by $e^{-2\mu\theta}$ and integrating, we get

$$v^2.e^{-2\mu\theta}=C^2 \quad \text{or} \quad v=Ce^{\mu\theta}.$$

Now $\quad r=ae^{\theta\cot\alpha}$. $\quad\therefore\quad \left(\frac{r}{a}\right)^{1/\cot\alpha}=e^{\theta}$

or $$\left(\frac{r}{a}\right)^{\mu\tan\alpha}=e^{\mu\theta}.$$

$$\therefore \quad v=C\left(\frac{r}{a}\right)^{\mu\tan\alpha}.$$

Initially when $r=a$, $v=-V$ (towards the pole). $\therefore$ $-V=C$.

$$\therefore \quad v=\left(\frac{r}{a}\right)^{\mu\tan\alpha}(-V) \quad \text{or} \quad v=V\cdot\left(\frac{r}{a}\right)^{\mu\tan\alpha}$$

or $$\frac{ds}{dt}=-V\frac{r^{\mu\tan\alpha}}{a^{\mu\tan\alpha}} \quad \text{or} \quad \frac{ds}{dr}\cdot\frac{dr}{dt}=-V\cdot\frac{r^{\mu\tan\alpha}}{a^{\mu\tan\alpha}}.$$

But $\cos\phi=\frac{dr}{ds}$ or $\cos\alpha=\frac{dr}{ds}$, $\because$ $\phi=\alpha$.

$$\therefore \quad \frac{1}{\cos\alpha}\cdot\frac{dr}{dt}=\frac{-V}{a^{\mu\tan\alpha}}\cdot r^{\mu\tan\alpha}$$

or $$-\frac{a^{\mu\tan\alpha}}{V\cos\alpha}\int_a^0 r^{-\mu\tan\alpha}\,dr=\int_0^t dt$$

or $$-\frac{a^{\mu \tan \alpha}}{V \cos \alpha}\left[\frac{r^{-\mu \tan \alpha+1}}{-\mu \tan \alpha+1}\right]_a^0=t.$$

$$\therefore \quad t=-\frac{a^{\mu \tan \alpha}}{V \cos \alpha}\cdot\frac{0-a^{-\mu \tan \alpha+1}}{-\mu \tan \alpha+1}$$

$$=\frac{a^{\mu \tan \alpha-\mu \tan \alpha+1}}{V \cos \alpha}\cdot\frac{\cos \alpha}{\cos \alpha-\mu \sin \alpha}$$

$$=\frac{a}{V}\cdot\frac{1}{\cos \alpha-\mu \sin \alpha}.$$ Proved.

§ 4. Motion on a cycloid.

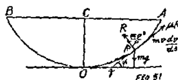
The base of a rough cycloidal arc is horizontal and vertex downwards. Discuss the motion of a bead down the arc.

(Dehli Hons. 63 ; Sagar 63 ; Agra 46, 48, 51, 57 ; Rajputana 59, 61, 62 ; Cal. Hons. 58)

Intrinsic equation of a cycloid is $s=4a \sin \psi$.

$\rho=\frac{ds}{d\psi}=4a \cos \psi$. Its length $=8a$. Also $\psi=\frac{\pi}{2}$ for cusp and $\psi=0$ at the vertex.

Since the particle is sliding down the arc the force of friction acts up. Hence we have the following equations of motion :—



$$mv\frac{dv}{ds}=\mu R-mg \sin \psi \ ; \qquad ...(1)$$

$$m\frac{v^2}{\rho}=R-mg \cos \psi. \qquad ...(2)$$

Eliminating R between (1) and (2), we get

$$mv\frac{dv}{ds}=\mu\left(m\frac{v^2}{\rho}+mg \cos \psi\right)-mg \sin \psi$$

or $$\frac{1}{2}\frac{dv^2}{ds}=\mu\frac{v^2}{\rho}+g\,(\mu \cos \psi-\sin \psi).$$

Multiplying both sides by 2ρ, we get

$$\frac{dv^2}{ds}\cdot\frac{ds}{d\psi}-2\mu v^2=2g\cdot 4a\cos\psi\,(\mu\cos\psi-\sin\psi),$$

$$\therefore\quad \rho=\frac{ds}{d\psi}=4a\cos\psi$$

or $$\frac{dv^2}{d\psi}-2\mu v^2=8ag\cos\psi\,(\mu\cos\psi-\sin\psi). \qquad \ldots(3)$$

Above is a linear differential equation and clearly I.F. is $e^{-2\mu\psi}$. Multiplying both sides by I. F. and integrating, we get

$$v^2.e^{-2\mu\psi}=8ag\int e^{-2\mu\psi}.\cos\psi\,(\mu\cos\psi-\sin\psi)\,d\psi+C. \qquad \ldots(4)$$

Put $\quad e^{-\mu\psi}(\mu\cos\psi-\sin\psi)=z.$ **(Important)**

$$\therefore\quad [-\mu e^{-\mu\psi}(\mu\cos\psi-\sin\psi)$$
$$+e^{-\mu\psi}(-\mu\sin\psi-\cos\psi)]\,d\psi=dz$$

or $$-e^{-\mu\psi}[1+\mu^2]\cos\psi\,d\psi=dz.$$

$$\therefore\quad e^{-\mu\psi}.\cos\psi\,d\psi=-\frac{dz}{1+\mu^2}. \qquad \ldots(5)$$

Hence from (4) and (5), we get

$$v^2e^{-2\mu\psi}=8ag\int e^{-\mu\psi}.\cos\psi.e^{-\mu\psi}(\mu\cos\psi-\sin\psi)\,d\psi+C$$

$$=8ag\int z.\frac{-dz}{1+\mu^2}+c$$

$$=-\frac{8ag}{1+\mu^2}.\frac{z^2}{2}+c\ \text{[put for } z]$$

or $$v^2e^{-2\mu\psi}=-\frac{4ag}{1+\mu^2}e^{-2\mu\psi}(\mu\cos\psi-\sin\psi)^2+C. \qquad \ldots(6)$$

In all the questions to follow we shall have to follow the procedure as above and arrive at result (6). The constant of integration C is found from initial conditions provided by each question separately.

The substitution $e^{-\mu\psi}(\mu\cos\psi-\sin\psi)=z$ should be carefully remembered. Having found the constant of

integration, we can find the value of v^2 from (6) and putting for v^2 in (2), we shall find the value of R.

Ex. 19. *A particle slides in a vertical plane down a rough cycloidal arc whose axis is vertical and vertex downwards starting from a point where the tangent makes an angle* θ *with the horizon and coming to rest at the vertex, show that* $\mu e^{\mu\theta}=\sin\theta-\mu\cos\theta$. (Sagar 63; Agra 45, 51, 56, 60, 65)

Proceeding exactly as in § 4, we arrive at result (6).

$$v^2e^{-2\mu\psi}=-\frac{4ag}{1+\mu^2}e^{-2\mu\psi}(\mu\cos\psi-\sin\psi)^2+C.$$

We are given that $v=0$ when $\psi=\theta$.

$$\therefore \quad C=\frac{4ag}{1+\mu^2}e^{-2\mu\theta}(\mu\cos\theta-\sin\theta)^2.$$

$$\therefore \quad v^2=\frac{4ag}{(1+\mu^2)}e^{-2\mu\theta}(\mu\cos\theta-\sin\theta)^2$$

$$-\frac{4ag}{1+\mu^2}e^{-2\mu\psi}(\mu\cos\psi-\sin\psi)^2. \qquad \ldots(1)$$

Again we are given that it comes to rest at the vertex where $\psi=0$.

Hence putting $v=0$ and $\psi=0$ in (1), we get

$$\frac{4ag}{1+\mu^2}e^{-2\mu\theta}(\mu\cos\theta-\sin\theta)^2=\frac{4ag}{1+\mu^2}(\mu)^2.$$

Cancel $\frac{4ag}{1+\mu^2}$ from both sides and taking square root,

we get $\qquad \pm e^{-\mu\theta}(\mu\cos\theta-\sin\theta)=\mu$

or $\qquad \mu e^{\mu\theta}=\pm(\mu\cos\theta-\sin\theta)$.

Taking $-$ive sign, we get

$$\mu e^{\mu\theta}=\sin\theta-\mu\cos\theta. \qquad \text{Proved.}$$

Ex. 20. *The base of a rough cycloidal arc is horizontal and its vertex downwards. A bead slides along it starting from rest at the cusp and coming to rest at the vertex. Show that* $\mu^2e^{\mu\pi}=1$.

(Indore 62, 66; Cal. Hons. 60; Punjab 56; Agra 59, 61, 66)

Proceeding exactly as in § 4, we have result (

$$v^2 e^{-2\mu\psi} = -\frac{4ag}{1+\mu^2} e^{-2\mu\psi} (\mu \cos \psi - \sin \psi)^2 + C.$$

We are given that $v=0$, when $\psi=\frac{\pi}{2}$ at cusp. Also $v=0$ when $\psi=0$ at the vertex.

$$\therefore \quad 0=\frac{-4ag}{1+\mu^2} e^{-\mu\pi} (0-1)^2+C \text{ for cusp,}$$

$$0=\frac{-2ag}{1+\mu^2} . 1 \, (\mu-0)^2+C \text{ for vertex.}$$

From above by subtracting, we get

$$e^{-\mu\pi} . 1=\mu^2 \quad \text{or} \quad \mu^2 e^{\mu\pi}=1. \qquad \textbf{Proved.}$$

Ex. 21. *A rough cycloid has its plane vertical and the line joining its cusps horizontal. A heavy particle slides down the curve from rest at a cusp and comes to rest again at the point on the other side of the vertex where the tangent is inclined at 45° to the vertical. Show that the coefficient of friction satisfies the equation $3\mu\pi+4 \log (1+\mu)=2 \log 2$.*

(Sagar 64, Agra 56, Vikram 63)

Proceeding as in § 4, we have the equation (6) as

$$v^2 e^{-2\mu\psi}=\frac{-4ag}{1+\mu^2} e^{-2\mu\psi} (\mu \cos \psi-\sin \psi)^2+C.$$

We are given that $v=0$, when $\psi=\frac{\pi}{2}$ at cusp. Also $v=0$, when $\psi=-45°$ as we are given that tangent is inclined at 45° to the vertical and hence at 45° to the horizontal and since it is on the other side of the vertex, ψ is measured in clockwise direction and hence we have taken $\psi=-45°$.

$$\therefore \quad 0=\frac{-4ag}{1+\mu^2} e^{-\mu\pi} (0-1)^2+C \qquad \text{for } \psi=\frac{\pi}{2}$$

$$0=\frac{-4ag}{1+\mu^2} e^{\frac{1}{2}\pi\mu} \left(\mu . \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)^2+C \quad \text{for } \psi=-\frac{\pi}{4}.$$

Subtracting, we get

$$e^{-\mu\pi}\,.\,1=e^{\frac{1}{2}\pi\mu}\,\frac{(\mu+1)^2}{2}$$

or
$$2=e^{\frac{3}{2}\mu\pi}\,(\mu+1)^2.$$

Taking log of both sides, we get

$$\log 2=\frac{3\pi}{2}\,\mu+2\log(\mu+1)$$

or
$$2\log 2=3\mu\pi+4\log(1+\mu). \qquad \text{Proved.}$$

Ex. 22. *A particle starts from rest from the cusp of a rough cycloid whose axis is vertical and vertex downwards. Show that its velocity at the vertex is to its velocity at the same point when the cycloid is smooth as*

$$(e^{-\mu\pi}-\mu^2)^{1/2} : \sqrt{(1+\mu^2)},$$

(Cal. Hon's 58 ; Rajputana 63)

where μ is the coefficient of friction. Further show that the particle will certainly come to rest before reaching the vertex if the coefficient of friction be ·5, having given that

log 2=·69315. (Punjab 57 ; Agra 62)

Proceeding exactly as in § 4, we have

$$v^2e^{-2\mu\psi}=\frac{-4ug}{1+\mu^2}\,e^{-2\mu\psi}\,(\mu\cos\psi-\sin\psi)^2+C.$$

Since the particle starts from rest at the cusp when $\psi=\frac{\pi}{2}$, we have

$$0=\frac{-4ag}{1+\mu^2}\,e^{-\mu\pi}\,(0-1)^2+C.$$

$$\therefore\quad C=\frac{4ag}{1+\mu^2}\,e^{-\mu\pi}\,.\,1.$$

$$\therefore\quad v^2e^{-2\mu\psi}=\frac{4ag}{1+\mu^2}\,e^{-\mu\pi}-\frac{4ag}{1+\mu^2}\,e^{-2\mu\psi}\,(\mu\cos\psi-\sin\psi)^2.$$

If v_1 be the velocity at the vertex where $\psi=0$, we have from above by putting $v=v_1$ and $\psi=0$.

$$v_1^2 \cdot 1 = \frac{4ag}{1+\mu^2} e^{-\mu\pi} - \frac{4ag}{1+\mu^2} \cdot 1 \cdot (\mu \cdot 1 - 0)^2$$

or $$v_1^2 = \frac{4ag}{1+\mu^2} [e^{-\mu\pi} - \mu^2]. \quad ...(1)$$

If v_2 be the velocity at the vertex when the cycloid is smooth, then putting $\mu=0$ in (1), we get

$$v_2^2 = \frac{4ag}{1} [1-0] = 4ag. \quad ...(2)$$

Hence from (1) and (2), we get

$$\frac{v_1^2}{v_2^2} = \frac{e^{-\mu\pi} - \mu^2}{1+\mu^2}$$

or $$v_1 : v_2 = \sqrt{(e^{-\mu\pi} - \mu^2)} : \sqrt{(1+\mu^2)}.$$

Again if the particle comes to rest at the lowest point *i.e.* vertex, then $v_1=0$ and hence from (1), $e^{-\mu\pi} - \mu^2 = 0$ or $\mu^2 e^{\mu\pi} = 1$. Hence it will come to rest before reaching the vertex if $\mu^2 e^{\mu\pi} > 1$ or $\mu e^{\mu\pi/2} > 1$

or if $\log \mu + \frac{\mu\pi}{2} \log e > \log 1$, *i.e.* > 0 *i.e.* +ive

or $\log \frac{1}{2} + \frac{1}{4} \cdot \frac{22}{7} \cdot 1 > 0$, $\because$ $\cdot 5 = \frac{1}{2}$ and $\pi = \frac{22}{7}$

or if $\frac{22}{28} - \log 2 > 0$ or $\frac{22}{28} - \cdot 69315 > 0$

which is true as $\frac{22}{28} = \cdot 785$.

Therefore if $\mu = \cdot 5$, the particle will come to rest before reaching the vertex.

Ex. 23. *A particle starts from rest from the cusp of a rough cycloid whose axis is vertical and vertex downwards. Show that it will come to rest before reaching the vertex if* $\mu e^{\pi\mu/2} > 1$ *or* $e^{\frac{1}{2}\pi \tan\lambda} > \cot\lambda$ *where* λ *is the angle of friction.*

Refer result (1) of Q. 22.. We know that $\tan\lambda = \mu$.

$$e^{-\mu\pi} - \mu^2 < 0 \quad \text{or} \quad e^{-\mu\pi} < \mu^2 \quad \text{or} \quad 1 < \mu^2 e^{\mu\pi}$$

or $$\mu e^{\mu\pi/2} > 1 \quad \text{or} \quad e^{\frac{1}{2}\pi \tan\lambda} > \cot\lambda.$$

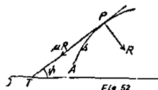
Ex. 24. *A bead moves along a rough curved wire which is such that it changes its direction of motion with constant*

angular velocity. Show that a possible form of wire is an equiangular spiral. (Agra 40)

The equations of motion of the bead are clearly

$$mv\frac{dv}{ds}=-\mu R \quad \ldots(1)$$

and $$m\frac{v^2}{\rho}=R. \quad \ldots(2)$$



Eliminating R, we get

$$mv\frac{dv}{ds}=-\mu . m\frac{v^2}{\rho} \quad \text{or} \quad \frac{dv}{ds}\cdot\frac{ds}{d\psi}=-\mu v \quad \text{or} \quad \frac{dv}{d\psi}=-\mu v$$

or $$\frac{dv}{v}=-\mu\, d\psi.$$

Integrating, $\log v=-\mu\psi+\log k$ or $\log\frac{v}{k}=-\mu\psi$

or $$v=ke^{-\mu\psi} \quad \text{or} \quad \frac{ds}{dt}=ke^{-\mu\psi}$$

or $$\frac{ds}{d\psi}\cdot\frac{d\psi}{dt}=ke^{-\mu\psi}.$$

We are given that it changes its direction of motion ψ with constant angular velocity.

$$\therefore \quad \frac{d\psi}{dt}=\text{constant}=c.$$

$$\therefore \quad \frac{ds}{d\psi}.c=ke^{-\mu\psi}$$

or $$c\, ds=ke^{-\mu\psi}.d\psi.$$

Integrating, we get $cs=-\frac{k}{\mu}e^{-\mu\psi}+p$

or $$s=-\frac{k}{\mu c}e^{-\mu\psi}+\frac{p}{c}.$$

Above is of the form $s=a+be^{-\mu\psi}$ which represents the intrinsic equation of equiangular spiral.

Ex. 25. *A small bead is threaded on a rough rigid wire in the form of an equiangular spiral of angle α. The bead is projected away from the pole with any velocity. Prove that the intervals of time between the successive instants at which the bead is moving in the same direction as at first form a G. P. of common ratio $e^{2\pi(\mu+\cot\alpha)}$*

We have $mv\dfrac{dv}{ds}=-\mu R$ and $\dfrac{mv^2}{\rho}=R$ and proceeding exactly as in Ex. 24, we have

$$v=\frac{ds}{dt}=ke^{-\mu\psi}$$

or
$$\frac{ds}{d\psi}\cdot\frac{d\psi}{dt}=ke^{-\mu\psi}. \quad ...(1)$$

Now the equation of equiangular spiral of angle α is $r=ae^{\theta\cot\alpha}$ and its pedal equation is $p=r\sin\alpha$ and $\phi=\alpha$ and $\dfrac{ds}{d\psi}=\rho=r.\dfrac{dr}{dp}=\dfrac{r}{\sin\alpha}=r\operatorname{cosec}\alpha.$ Also $\psi=\theta+\phi=\theta+\alpha.$

Hence from (1), we get

$$\frac{r}{\sin\alpha}\cdot\frac{d\psi}{dt}=ke^{-\mu\psi}.$$

or
$$\frac{d\psi}{dt}=\frac{k\sin\alpha.e^{-\mu\psi}}{r}=\frac{k\sin\alpha.e^{-\mu\psi}}{ae^{\theta\cot\alpha}}$$

or
$$\frac{d\psi}{dt}=\frac{k\sin\alpha}{a}.e^{-\mu\psi}.e^{-(\psi-\alpha)\cot\alpha} \quad \because \quad \psi=\theta+\alpha$$

or
$$\frac{d\psi}{dt}=\frac{k\sin\alpha}{a}.e^{\alpha\cot\alpha}.e^{-(\mu+\cot\alpha)\psi}$$

or
$$e^{(\mu+\cot\alpha)\psi}\,d\psi=\frac{k\sin\alpha}{a}e^{\alpha\cot\alpha}.dt.$$

Integrating, $\dfrac{e^{(\mu+\cot\alpha)\psi}}{\mu+\cot\alpha}=\dfrac{k\sin\alpha}{a}e^{\alpha\cot\alpha}.t+A$

or since α, k, a and μ are constants, above is of the form

$$e^{(\mu+\cot\alpha)\psi}=Bt+C. \quad ...(2)$$

Let initially, when $t=0, \psi=\psi_0$,

$$\therefore \quad C=e^{(\mu+\cot\alpha)\psi_0}. \quad \ldots(3)$$

Since we are to find the intervals, when the particle is moving in the same direction as at first, hence we give to ψ the values $\psi_0+2\pi$, $\psi_0+4\pi$, $\psi_0+6\pi$ and so on and let the corresponding values of t be t_1, t_2, t_3 and so on and we are to find the values of t_2-t_1, t_3-t_2 and so on.

Putting the above data in (2), we get

$$e^{(\mu+\cot\alpha)(\psi_0+2\pi)}=Bt_1+C$$

or $$e^{(\mu+\cot\alpha)\psi_0}\cdot e^{(\mu+\cot\alpha)2\pi}-C=Bt_1$$

or $$C\{e^{(\mu+\cot\alpha)2\pi}-1\}=Bt_1 \text{ by (3)}.$$

Similarly $$C\{e^{(\mu+\cot\alpha)4\pi}-1\}=Bt_2,$$

$$C\{e^{(\mu+\cot\alpha)6\pi}-1\}=Bt_3 \text{ and so on.}$$

$$\therefore \quad t_2-t_1=\frac{C}{B}e^{(\mu+\cot\alpha)2\pi}[e^{(\mu+\cot\alpha)2\pi}-1],$$

$$t_3-t_2=\frac{C}{B}e^{(\mu+\cot\alpha)4\pi}[e^{(\mu+\cot\alpha)2\pi}-1]$$

Clearly above intervals of time t_2-t_1, t_3-t_2, ...form a G. P. whose common ratio is $e^{(\mu+\cot\alpha)2\pi}$.

Ex. 26. *A curve in a vertical plane is such that the time of describing any arc, measured from a fixed point O, is equal to the time of sliding down the chord of the arc. Show that the curve is a lemniscate of Bernoulli whose node is at O and whose axis is inclined at 45° to the vertical.*

(Indore 66 ; Raj. 54 ; Agra 40, 54)

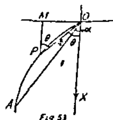


Fig 51

Let OPA be an arc of the curve whose equation may be taken as $r=f(\theta)$ referred to O as pole and a vertical through O as initial line. If the radius vector OA be inclined at an angle α to initial line, then from $r=f(\theta)$, we have $OA=f(\alpha)$. ...(1)

The particle will slide down the chord OA with acceleration $g\sin(90-\alpha)$ *i.e.* $g\cos\alpha$. If t be the time, then using $s=\frac{1}{2}ft^2$, we have

$$OA=\tfrac{1}{2}g\cos\alpha . t^2;\quad \therefore\quad t=\sqrt{\left[\frac{2.f(\alpha)}{g\cos\alpha}\right]}. \text{ by (1)} \quad ...(2)$$

Again if v be the velocity at P, then from the equation of energy, we have

change in K. E. = work done by gravity in falling $MP=r\cos\theta$.

$$\therefore\quad \tfrac{1}{2}mv^2=mg.r\cos\theta$$

or $$v^2=2gr\cos\theta=2g\cos\theta.f(\theta)$$

or $$\left(\frac{dr}{dt}\right)^2+\left(r\frac{d\theta}{dt}\right)^2=2g\cos\theta.f(\theta)$$

or $$\left(\frac{dr}{d\theta}.\frac{d\theta}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2=2g\cos\theta.f(\theta)$$

or $$\{f'^2(\theta)+f^2(\theta)\}\left(\frac{d\theta}{dt}\right)^2=2g\cos(\theta).f(\theta).\quad \because r=f(\theta).$$

or $$\int_0^t dt=\int^\alpha\sqrt{\left[\frac{f^2(\theta)+f'^2(\theta)}{2g\cos\theta.f(\theta)}\right]}d\theta$$

or $$t=\int^\alpha\sqrt{\left[\frac{f^2(\theta)+f'^2(\theta)}{2g\cos\theta.f(\theta)}\right]}\,d\theta=\sqrt{\left[\frac{2f(\alpha)}{g\cos\alpha}\right]} \text{ by (2).}$$

Differentiating both sides w. r. t. α, we get

$$\sqrt{\left[\frac{f^2(\alpha)+f'^2(\alpha)}{2g\cos\alpha f(\alpha)}\right]}=\frac{1}{2}\sqrt{\left[\frac{g\cos\alpha}{2f(\alpha)}\right]}.\frac{2}{g}\left[\frac{\cos\alpha.f'(\alpha)+f(\alpha)\sin\alpha}{\cos^2\alpha}\right].$$

Squaring both sides, we get

$$\frac{f^2(\alpha)+f'^2(\alpha)}{2g\cos\alpha f(\alpha)}=\frac{1}{g}.\frac{\cos\alpha}{2f(\alpha)}$$
$$\times\left[\frac{\cos^2\alpha.f'^2(\alpha)+\sin^2\alpha.f^2(\alpha)+2\sin\alpha\cos\alpha.f(\alpha).f'(\alpha)}{\cos^4\alpha}\right]$$

or $$f^2(\alpha)\cos^2(\alpha)+f'^2(\alpha)\cos^2(\alpha)=\cos^2\alpha f'^2(\alpha)+\sin^2\alpha f^2(\alpha)$$
$$+\sin 2\alpha.f'(\alpha).f(\alpha)$$

or $$(\cos^2\alpha-\sin^2\alpha).f^2(\alpha)=\sin 2\alpha.f'(\alpha).f(\alpha)$$

or $$\frac{\cos 2\alpha}{\sin 2\alpha}=\frac{f'(\alpha)}{f(\alpha)}.$$

Integrating both sides, we get

$$\tfrac{1}{2}\log \sin 2\alpha = \log f(\alpha) - \log a$$

or $$\log \sqrt{(\sin 2\alpha)} = \log \frac{f(\alpha)}{a}$$

$$\therefore \quad f(\alpha) = a\sqrt{(\sin 2\alpha)}.$$

Therefore the equation of the curve is the locus of point α given by $f(\theta) = a\sqrt{(\sin 2\alpha)}$ or $r = a\sqrt{(\sin 2\theta)}$

or $$r^2 = a^2 \sin 2\theta \quad \text{or} \quad r^2 = a^2 \cos\left(2\theta - \frac{\pi}{2}\right)$$

or $$r^2 = a^2 \cos 2\left(\theta - \frac{\pi}{4}\right)$$

which represents lemniscate of Bernoulli, whose axis is inclined at an angle of 45° to the initial line, which we have taken to be vertical in question.

Ex. 27. *A rough parabolic wire with latus rectum 4 a is placed with its axis vertical and vertex downwards and a bead is projected along it from the lowest point with velocity u. Show that the bead will come to rest at a distance $\frac{a}{n^2}$ from the focus*

where $$\mu \cos^{-1} n = \log\left\{ n\sqrt{\left(1 + \frac{u^2}{2ag}\right)}\right\}.$$



Taking SX as the initial line, the polar equation of the parabola is

$$\frac{2a}{r} = 1 + \cos\theta \quad \text{or} \quad r = a\sec^2\frac{\theta}{2}.$$

$$\therefore \quad \frac{dr}{d\theta} = 2a \sec\frac{\theta}{2}\sec\frac{\theta}{2}\tan\frac{\theta}{2}\cdot\frac{1}{2}$$

or $$\frac{1}{r}\frac{dr}{d\theta} = \tan\frac{\theta}{2} \quad \text{or} \quad \cot\phi = \tan\frac{\theta}{2}.$$

$$\therefore \quad \phi = 90 - \frac{\theta}{2} \quad \text{and} \quad \psi = \theta + \phi = 90 + \frac{\theta}{2}. \qquad \ldots(1)$$

Also $p = r\sin\phi = r\cos\frac{\theta}{2} = r\sqrt{\left(\frac{a}{r}\right)} = \sqrt{a}.\sqrt{r}.$

$$\therefore \quad \frac{dp}{dr} = \frac{\sqrt{a}}{2\sqrt{r}}. \quad \therefore \quad \rho = r\frac{dr}{dp} = 2r\sqrt{\left(\frac{r}{a}\right)} = 2a\sec^3\frac{\theta}{2}. \quad \ldots(2)$$

Also when $r=\frac{a}{n^2}$, then $a\sec^2\frac{\theta}{2}=\frac{a}{n^2}$ or $\cos\frac{\theta}{2}=n$.

$$\therefore \quad \frac{\theta}{2}=\cos^{-1} n. \qquad \ldots(3)$$

The equations of motion are

$$mv\frac{dv}{ds}=-\mu R-mg\cos\phi=-\mu R-mg\sin\frac{\theta}{2} \text{ by (1)}$$

and $\quad \frac{mv^2}{\rho}=R-mg\sin\phi=R-mg\cos\frac{\theta}{2}.$

Eliminating R, we get

$$mv\frac{dv}{ds}=-\mu\left(\frac{mv^2}{\rho}+mg\cos\frac{\theta}{2}\right)-mg\sin\frac{\theta}{2}$$

or $\quad \frac{1}{2}\frac{dv^2}{ds}+\frac{\mu v^2}{\rho}=-g\left(\mu\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right).$

Multiply both sides by 2ρ and put $\rho=2a\sec^3\theta/2$ by (2)

or $\quad \frac{dv^2}{ds}\cdot\frac{ds}{d\psi}+2\mu v^2=-2g\,.\,2a\sec^3\frac{\theta}{2}\left(\mu\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)$

or $\quad \frac{dv^2}{d\theta}\cdot\frac{d\theta}{d\psi}+2\mu v^2=-4ag\left(\mu\sec^2\frac{\theta}{2}+\tan\frac{\theta}{2}\sec^3\frac{\theta}{2}\right)$

or $\quad \frac{dv^2}{d\theta}\,.\,2+2\mu v^2=-4ag\left(\mu\sec^2\frac{\theta}{2}+\tan\frac{\theta}{2}\sec^2\frac{\theta}{2}\right)$

$$\because \quad \psi=90+\frac{\theta}{2}$$

or $\quad \frac{dv^2}{d\theta}+\mu v^2=-2ag\left(\mu\sec^2\frac{\theta}{2}+\tan\frac{\theta}{2}\sec^2\frac{\theta}{2}\right).$

Above is linear equation and I. F.$=e^{\mu\theta}$. Multiplying both sides by $e^{\mu\theta}$ and integrating, we get

$$v^2\,.\,e^{\mu\theta}=-2ag\int e^{\mu\theta}\left(\tan\frac{\theta}{2}\sec\frac{\theta}{2}\,.\sec\frac{\theta}{2}+\mu\sec^2\frac{\theta}{2}\right)d\theta+C$$

$$=-2ag\left[e^{\mu\theta}\sec^2\frac{\theta}{2}-\int e^{\mu\theta}\,.\,\mu\sec^2\frac{\theta}{2}\,d\theta\right.$$

$$\left.+\int e^{\mu\theta}\,\mu\sec^2\frac{\theta}{2}\,d\theta\right]+C$$

or $\quad v^2 e^{\mu\theta} = -2age^{\mu\theta} \sec^2 \frac{\theta}{2} + C.$

Initially for the vertex $\theta = 0$ and $v = u$.

$\therefore \quad u^2 = -2ag + C \quad$ or $\quad C = u^2 + 2ag.$

$\therefore \quad v^2 e^{\mu\theta} = -2age^{\mu\theta} \sec^2 \frac{\theta}{2} + u^2 + 2ag.$

The bead will come to rest when $v = 0$.

$\therefore \quad 2age^{\mu\theta} \sec^2 \frac{\theta}{2} = u^2 + 2ag \quad$ or $\quad e^{\mu\theta} \sec^2 \frac{\theta}{2} = 1 + \frac{u^2}{2ag}$

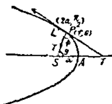
or $$e^{\mu\theta/2} = \cos \frac{\theta}{2} \sqrt{\left(1 + \frac{u^2}{2ag}\right)} = n \sqrt{\left(1 + \frac{u^2}{2ag}\right)} \text{ by (3).}$$

Taking log of both sides, we get

$$\frac{\mu\theta}{2} = \log \left[n \sqrt{\left(1 + \frac{u^2}{2ag}\right)} \right]$$

or $$\mu \cos^{-1} n = \log n \sqrt{\left(1 + \frac{u^2}{2ag}\right)} \text{ by (3).} \qquad \text{Proved.}$$

Ex. 28. *A smooth wire in the form of a parabola of latus rectum 4a is fixed in a horizontal plane. A small ring of mass m can slide on it and is attached to the focus by a light elastic string of natural length a and modulus λ. If the ring is held at the end of the latus rectum and released, show that its distance from the axis of the parabola at time t is* $2a \cos \left[t \sqrt{\left(\frac{\lambda}{4am} \right)} \right]$.



Refer Q. 5 P. 189. Let T be the tension in the string acting along PS, so

that $\quad T = \lambda \frac{r - a}{a}.$

Tangential equation of motion is

$$mv \frac{dv}{ds} = -T \cos \phi$$

or $\quad mv \frac{dv}{ds} = -\frac{\lambda}{a}(r - a) \frac{dr}{ds} \quad$ or $\quad v\, dv = -\frac{\lambda}{am}(r - a)\, dr.$

Integrating, $v^2 = \frac{-\lambda}{am}(r-a)^2 + A.$

Initially when $r=2a$ at the end of latus rectum, $v=0$.

$$\therefore \quad A = \frac{\lambda}{am} a^2.$$

$$\therefore \quad v^2 = \frac{\lambda}{am}[a^2-(r-a)^2] \; ; \quad \therefore \quad v = \sqrt{\left(\frac{\lambda}{am}\right)}[a^2-(r-a)^2]^{1/2}. \quad \ldots(1)$$

But $\frac{2a}{r} = 1+\cos\theta$ is the polar equation of parabola

or $$r = a\sec^2\frac{\theta}{2}.$$

$$r-a = a\left(\sec^2\frac{\theta}{2}-1\right) = a\tan^2\frac{\theta}{2}. \quad \text{Put in (1).}$$

$$\therefore \quad v = \sqrt{\left(\frac{\lambda}{am}\right)}.a\sqrt{\left(1-\tan^4\frac{\theta}{2}\right)}. \quad \ldots(2)$$

Again $$v = \frac{ds}{dt} = \frac{ds}{d\theta}\cdot\frac{d\theta}{dt}.$$

But $$\frac{ds}{d\theta} = \sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}$$

$$= \sqrt{\left(a^2\sec^4\frac{\theta}{2}+a^2\sec^4\frac{\theta}{2}\tan^2\frac{\theta}{2}\right)} = a\sec^3\frac{\theta}{2}.$$

$$\therefore \quad a\sec^3\frac{\theta}{2}\cdot\frac{d\theta}{dt} = \sqrt{\left(\frac{\lambda}{am}\right)}.a\sqrt{\left[\left(1+\tan^2\frac{\theta}{2}\right)\left(1-\tan^2\frac{\theta}{2}\right)\right]}$$

or $$\int_{\pi/2}^{0}\frac{\sec^2\frac{\theta}{2}}{\sqrt{\left(1-\tan^2\frac{\theta}{2}\right)}}\,d\theta = \int_0^t\sqrt{\left(\frac{\lambda}{am}\right)}\,dt.$$

Put $\tan\frac{\theta}{2} = z$; $\therefore$ $\frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta = dz$.

Also initally at the end of latus rectum,

$$\theta = \frac{\pi}{2} \; ; \quad \therefore \quad z = \tan\frac{\pi}{4} = 1.$$

$$\therefore \quad \int_1^z \frac{2\,dz}{\sqrt{(1-z^2)}}=\sqrt{\left(\frac{\lambda}{am}\right)}.t \quad \text{or} \quad 2\left[\sin^{-1}z\right]_1^z=\sqrt{\left(\frac{\lambda}{am}\right)}\,t$$

$$\text{or} \quad \sin^{-1}z-\frac{\pi}{2}=\tfrac{1}{2}\sqrt{\left(\frac{\lambda}{am}\right)}\,t \quad \text{or} \quad -\cos^{-1}z=\sqrt{\left(\frac{\lambda}{4am}\right)}\,t.$$

$$\therefore \quad z=\cos\sqrt{\left(\frac{\lambda}{4am}\right)}.t \quad \text{or} \quad \tan\frac{\theta}{2}=\cos\sqrt{\left(\frac{\lambda}{4am}\right)}.t \quad ...(3)$$

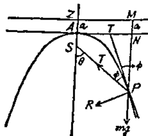
Now if r be the distance of particle from focus, then its distance from the axis is

$$r\sin\theta=a\sec^2\frac{\theta}{2}.2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=2a\tan\frac{\theta}{2}$$

$$=2a\cos\sqrt{\left(\frac{\lambda}{4am}\right)}\,t \text{ by (3).} \qquad \text{Proved.}$$

Ex. 29. *A heavy ring of mass m free to move on a smooth parabolic wire of latus rectum 4a with axis vertical and vertex upwards, is attached to the focus by an elastic string of natural length $\frac{a}{2}$ and modulus λ. The ring is projected from the vertex with velocity due to its depth below the directrix. Prove that it will come to rest at a point whose vertical depth below the vertex is $\frac{mga}{\lambda}$ after a time $\pi\sqrt{\left(\frac{am}{2\lambda}\right)}$.*

Just as in Q. 9 P. 205.



$$\frac{2a}{r}=1-\cos\theta=2\sin^2\frac{\theta}{2},$$

$$\therefore \quad r=a\,\text{cosec}^2\frac{\theta}{2}.$$

$$\text{Also} \quad \theta=2\phi \quad \text{or} \quad \phi=\frac{\theta}{2}.$$

$$T=\lambda.\frac{\text{extension}}{\text{natural length}}=\frac{\lambda\left(r-\frac{a}{2}\right)}{\frac{a}{2}}=\frac{\lambda\,(2r-a)}{a}.$$

The equation of motion along the tangent is

$$mv\frac{dv}{ds}=mg\cos\phi-T\cos\phi=\left[mg-\frac{\lambda(2r-a)}{a}\right]\frac{dr}{ds}.$$

$$\therefore\quad v\,dv=\frac{1}{am}\,[amg+a\lambda-2\lambda r]\,dr.$$

Integrating, $v^2=\dfrac{2}{am}\,[a(mg+\lambda)\,r-\lambda r^2]+A.$

Initially at the vertex where $r=a$, $v^2=(2ag)$ given, as the velocity at the vertex is that due to a fall from directrix to the vertex *i.e.* a distance a.

$$\therefore\quad 2ag=\frac{2}{am}\,[\,a(mg+\lambda)\,a-a^2\lambda]+A$$

or

$$2ag=\frac{2}{am}\cdot a^2mg+A\;;\quad\therefore\quad A=0.$$

$$\therefore\quad v^2=\frac{2}{am}\,[a(mg+\lambda)\,r-\lambda r^2]. \qquad ...(1)$$

The particle will come to rest when $v=0$.

$$\therefore\quad a(mg+\lambda)-\lambda r=0\;;\;\frac{amg}{\lambda}+a=r. \qquad ...(2)$$

Now r=distance of P from focus$=PS$

=distance of P from directrix *i.e.* PM

=distance of P from vertex *i. e.* $PN+a$.

$\therefore\quad \dfrac{amg}{\lambda}+a$=distance of P from vertex $PN+a$.

$\therefore$ distance of P from vertex$=\dfrac{amg}{\lambda}$. Proved.

Again $v=\dfrac{ds}{dt}=\dfrac{ds}{dr}\cdot\dfrac{dr}{dt}=\dfrac{1}{\cos\phi}\dfrac{dr}{dt}=\dfrac{1}{\cos\theta/2}\dfrac{dr}{dt}$, $\because\ \phi=\dfrac{\theta}{2}$

or $v=\dfrac{1}{\sqrt{\left(1-\frac{a}{r}\right)}}\dfrac{dr}{dt}$, $\because\ \sin\dfrac{\theta}{2}=\sqrt{\dfrac{a}{r}}$

or $$\sqrt{\left(\frac{2}{am}\right)}\sqrt{r}\left[a(mg+\lambda)-\lambda r\right]^{1/2}=\sqrt{\left(\frac{r}{r-a}\right)}\cdot\frac{dr}{dt} \text{ by (1)}$$

or $$\sqrt{\left(\frac{2\lambda}{am}\right)}(dt)=\frac{dr}{\sqrt{\left[(r-a)\left(\frac{a(mg+\lambda)}{\lambda}\right)-r\right]}}.$$

When $t=0$, $r=0$; when $t=t$ say, then

$$r=\frac{a\ (mg+\lambda)}{\lambda}=b \text{ say by (2).}$$

$$\therefore \int_0^t \sqrt{\left(\frac{2\lambda}{am}\right)}\, dt=\int_a^b \frac{dr}{\sqrt{(r-a)\ (b-r)}}.$$

Put $r=a\sin^2\theta+b\cos^2\theta$.

$\therefore$ $r-a=(b-a)\cos^2\theta$ so that when $r=a$, $\theta=\pi/2$.

$b-r=(b-a)\sin^2\theta$ so that when $r=b$, $\theta=0$.

Also $dr=2\,(a-b)\sin\theta\cos\theta\, d\theta$.

$$\therefore \sqrt{\left(\frac{2\lambda}{am}\right)}\cdot t=\int_{\pi/2}^0 -2\,d\theta=2\Big[\theta\Big]_0^{\pi/2}=\pi.$$

$$\therefore \quad t=\pi\sqrt{\left(\frac{am}{2\lambda}\right)}.$$

Ex. 30. *A bead of mass m is threaded on a smooth thin wire in the form of an equiangular spiral of angle* α *and is connected to the pole by an elastic string of modulus* λ *and natural length a; when the bead is at a distance a from the pole, it is projected with velocity* $\sqrt{\left(\frac{a\lambda}{m}\right)}$ *along the wire away from the pole. Show that the radial velocity of the bead when at distance r from the pole is* $\sqrt{\left[\left\{\frac{\lambda r\,(2a-r)}{am}\right\}\cos^2\alpha\right]}$ *and the length of the string is 2a after time* $\frac{\pi}{2}\sec\alpha\sqrt{\left(\frac{am}{\lambda}\right)}$.

We know that the equation of equiangular spiral is $r=ae^{\theta\cot\alpha}$ and its pedal equation is $p=r\sin\alpha$, *i. e.* $\phi=\alpha$.

Tension in the string is $\frac{\lambda}{a}(r-a)$ where a is its natural length. Tangential equation of motion is

$$mv\frac{dv}{ds}=-T\cos\phi=-\frac{\lambda}{a}(r-a)\frac{dr}{ds}$$

or $$v\,dv=-\frac{\lambda}{am}(r-a).$$

Integrating, we get

$$v^2=-\frac{\lambda}{am}(r-a)^2+A. \quad \text{Initially when } r=a,\ v^2=\frac{a\lambda}{m}.$$

$$\therefore\ A=\frac{a\lambda}{m};\quad \therefore\ v^2=\frac{a\lambda}{m}-\frac{\lambda}{am}(r-a)^2$$

or $$v=\sqrt{\left(\frac{\lambda}{am}\right)}\sqrt{\{a^2-(r-a)^2\}}.$$

But $$v=\frac{ds}{dt}=\frac{ds}{dr}\cdot\frac{dr}{dt}=\frac{1}{\cos\phi}\cdot\frac{dr}{dt}=\frac{1}{\cos\alpha}\cdot\frac{dr}{dt}.$$

$$\therefore\ \frac{1}{\cos\alpha}\cdot\frac{dr}{dt}=\sqrt{\left(\frac{\lambda}{am}\right)}\cdot\sqrt{[a^2-(r-a)^2]} \qquad ...(1)$$

or radial velocity $=\frac{dr}{dt}=\cos\alpha\,.\sqrt{\left(\frac{\lambda}{am}\right)}\sqrt{(2ar-r^2)}$

$$=\left[\frac{\lambda}{am}\,r\,(2a-r)\cos^2\alpha\right]^{1/2}. \qquad \text{Proved.}$$

Again from (1), $\int_a^{2a}\frac{dr}{\sqrt{\{a^2-(r-a)^2\}}}=\int_{t=0}^{t}\sqrt{\left(\frac{\lambda}{am}\right)}\cos\alpha.$

$\because$ when $t=0$, $r=a$ and when $t=t$ say, $r=2a$.

$$\left[\sin^{-1}\frac{r-a}{a}\right]_a^{2a}=\sqrt{\left(\frac{\lambda}{am}\right)}\cos\alpha\,.\,t$$

or $$\frac{\pi}{2}=\sqrt{\left(\frac{\lambda}{am}\right)}\cos\alpha\,.\,t\,;\quad \therefore\ t=\frac{\pi}{2}\sec\alpha\,.\sqrt{\left(\frac{am}{\lambda}\right)}.$$

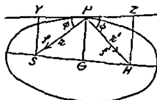
Ex. 31. *A particle P of unit mass describes an ellipse under an attraction f to focus S and attraction f' to the other focus H. If $SP=r$ and $HP=r'$, prove that*

$$\frac{1}{r^2}\frac{d}{dr}(r^2 f)=\frac{1}{r'^2}\frac{d}{dr'}(r'^2 f').$$

Hence show that if one force obeys the Newtonian Law, so also must obey the other. If the forces are equal, then each varies inversely as the product of focal distances of P. (Agra 59)

Note. Before giving the solution we shall establish the following two results. There is a force f at P towards focus S, where $SP=r$ and a force f' at P towards focus H where $HP=r'$.

We know that tangent and normal at any point on an ellipse are equally inclined to the focal radii SP and HP.



$\therefore \quad \angle SPY=\angle HPZ=\phi.$

$$\sin\phi=\frac{SY}{SP}=\frac{SY}{r} \text{ and } \sin\phi=\frac{HZ}{HP}=\frac{HZ}{r'}.$$

$$\therefore \quad \sin^2\phi=\frac{SY.HZ}{rr'}=\frac{b^2}{rr'}, \quad \because SY.HZ=b^2. \text{ [§ 5 P. 111]}$$

Hence $$\sin\phi=\frac{b}{\sqrt{(rr')}}. \qquad \text{...(1)}$$

Again we know that in an ellipse $x=a\cos t$, $y=b\sin t$.

$$\rho=\frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}=\frac{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}{ab}$$

$$=\frac{[a^2(1-\cos^2 t)+b^2\cos^2 t]^{3/2}}{ab}$$

$$=\frac{[a^2-(a^2-b^2)\cos^2 t]^{3/2}}{ab}=\frac{(a^2-a^2e^2\cos^2 t)^{3/2}}{ab}=\frac{(a^2-e^2x^2)^{3/2}}{ab}$$

$$=\frac{[(a-ex)(a+ex)]^{3/2}}{ab}=\frac{(SP.HP)^{3/2}}{ab}=\frac{(rr')^{3/2}}{ab}. \qquad \text{...(2)}$$

Also $SP+HP=2a$ or $r+r'=2a$.

Resolving the forces along the tangent, we have

$$v\frac{dv}{ds}=f\cos\phi-f'\cos\phi=-f\frac{dr}{ds}-f'\frac{dr'}{ds'}, \quad \because \cos\phi=\frac{dr}{ds}.$$

$$\therefore \quad \tfrac{1}{2} dv^2 = -(f\,dr + f'\,dr'). \qquad \ldots(3)$$

Resolving along the normal, we get

$$\frac{v^2}{\rho} = f \sin \phi + f' \sin \phi \quad [m=1]$$

or
$$v^2 = (f+f')\,\rho \sin \phi. \qquad \ldots(4)$$

Putting for v^2 from (4) in (3), we get

$$d\,[(f+f')\,\rho \sin \phi] = -2f\,dr - 2f'\,dr'$$

or $d\left[(f+f')\,\dfrac{(rr')^{3/2}}{ab} \cdot \dfrac{b}{\sqrt{(rr')}}\right] = -2f\,dr - 2f'\,dr'$ by (1) and (2)

or
$$\frac{1}{a}\,d\,[(f+f')rr'] = -2f\,dr - 2f'\,dr'$$

or
$$d\,(frr') + d\,(f'rr') = -2a\,(f\,dr + f'\,dr')$$

or
$$\frac{d}{dr}(frr')\,dr + \frac{d}{dr'}(f'rr')\,dr' = -2a\,(f\,dr + f'\,dr').$$

Now in an ellipse sum of the focal distances is constant and equal to $2a$.

$$\therefore \quad r + r' = 2a\,; \quad \therefore \quad dr = -dr'.$$

$$\therefore \quad \frac{d}{dr}(frr')\,dr - \frac{d}{dr'}(f'rr')\,dr = -2a\,(f\,dr - f'\,dr)$$

or
$$\frac{d}{dr}(frr') - \frac{d}{dr'}(f'rr') = -(r+r')\,(f-f')$$

$$\left(rr'\frac{df}{dr} + fr' + fr\,.\frac{dr'}{dr}\right) - \left(rr'\frac{df'}{dr'} + f'r + f'r'\,\frac{dr}{dr'}\right)$$
$$= -f\,(r+r') + f'\,(r+r').$$

Put $\quad \dfrac{dr}{dr'} = \dfrac{dr'}{dr} = -1.$

$$\therefore \quad \left(rr'\frac{df}{dr} + fr' - fr\right) - \left(rr'\frac{df'}{dr'} + f'r - f'r'\right)$$
$$= -f\,(r+r') + f'\,(r+r')$$

or
$$rr'\frac{df}{dr} + 2fr' = rr'\frac{df'}{dr'} + 2f'r.$$

Dividing throughout by rr', we get

$$\frac{df}{dr} + \frac{2f}{r} = \frac{df'}{dr'} + \frac{2f'}{r'}$$

or $$\frac{1}{r^2}\left(r^2\frac{df}{dr}+2r\cdot f\right)=\frac{1}{r'^2}\left(r'^2\frac{df'}{dr'}+2r'\cdot f'\right)$$

or $$\frac{1}{r^2}\cdot\frac{d}{dr}(r^2f)=\frac{1}{r'^2}\frac{d}{dr'}(r'^2f'). \qquad ...(4)$$

Relation (4) is the first part.

Now if f obeys Newtonian Law, then $f=\frac{\mu}{r^2}$ or $r^2f=\mu$, *i.e.* constant.

Hence from (4), we get $\frac{1}{r}\frac{d}{dr}(\mu)=\frac{1}{r'^2}\frac{d}{dr'}(r'^2f')$

or $0=\frac{1}{r'^2}\frac{d}{dr'}(r'^2f')$; $\therefore$ $r'^2f'=\text{const.}=\mu'$; $\therefore$ $f'=\frac{\mu'}{r'^2}$.

Hence it is clear that f' also obeys Newtonian Law and it proves the 2nd part.

We have proved that $f=\frac{\mu}{r^2}$ and also $f'=\frac{\mu'}{r'^2}$

$$\therefore \; ff'=\frac{\mu\mu'}{r^2r'^2}.$$

But if the forces be equal, *i.e.* $f'=f$, then

$$f^2=\frac{\mu\mu'}{r^2r'^2} \quad \text{or} \quad f=\frac{\sqrt{(\mu\mu')}}{rr'}=\frac{k}{rr'},$$

i.e. force varies inversely as the product of the focal distances of the point.

$v=\frac{dx}{dt}$

$\frac{d}{dt}(v)=\frac{dv}{dx}\cdot\frac{dx}{dt}=v\frac{dv}{dx}$

$\frac{d^2x}{dt^2}=v\frac{dv}{dx}=\frac{dv}{dt}$

CHAPTER

MOTION IN A RESISTING MEDIUM

Vertically downwards motion.

§ 1. *A particle falls under gravity (supposed constant) in a resisting medium whose resistance varies as the square of the velocity ; to find the motion if the particle starts from rest.*

(Osmania 66 ; Delhi Hons. 58 ; Vikram 63 ; Agra 47, 53, 58, 67)

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motion. Also the force due to gravity is mg acting vertically downwards. Therefore using $P=mf$, we have the equation of motion of the particle as

$$m\frac{d^2x}{dt^2}=mg-mkv^2$$

or
$$\frac{d^2x}{dt^2}=g\left(1-\frac{k}{g}v^2\right). \qquad ...(A)$$

If V be the velocity of the particle when its acceleration is zero, then
$$0=g\left(1-\frac{k}{g}V^2\right) \text{ or } \frac{k}{g}=\frac{1}{V^2}. \qquad ...(B)$$

The motion would then be unresisted (acceleration being zero) and the velocity of the particle would continue to be V. It is on account of this fact that V is called the terminal velocity $=\sqrt{\left(\frac{g}{k}\right)}$.

Velocity and distance relation.

Now from (A) and (B), we have

$$\frac{d^2x}{dt^2}=g\left(1-\frac{v^2}{V^2}\right) \text{ or } v\frac{dv}{dx}=\frac{g}{V^2}(V^2-v^2). \qquad ...(1)$$

or $$\frac{2g}{V^2}\,dx=\frac{2v}{V^2-v^2}\,dv.$$

Integrating, we get $\frac{2g}{V^2}x=-\log(V^2-v^2)+C.$

Initially when $x=0$, $v=0$; $\therefore$ $C=\log V^2$.

$$\therefore\ \frac{2g}{V^2}x=\log V^2-\log(V^2-v^2)$$

or $$\frac{2g}{V^2}x=\log\frac{V^2}{V^2-v^2}$$

or $$\frac{V^2}{V^2-v^2}=e^{\frac{2g}{V^2}x}\quad\text{or}\quad\frac{V^2-v^2}{V^2}=e^{-\frac{2g}{V^2}x}$$

or $$v^2=V^2\left(1-e^{-\frac{2g}{V^2}x}\right). \qquad ...(2)$$

(Cal. Hons. 63)

Above relation gives us the velocity of the particle at any distance x.

K. E. is $\qquad \frac{1}{2}mv^2=\frac{1}{2}mV^2\left(1-e^{-\frac{2g}{V^2}x}\right).$

Time and velocity relation.

In order to find the relation between v and t, we write

(1) as $\qquad \frac{dv}{dt}=\frac{g}{V^2}(V^2-v^2)\quad$ or $\frac{dv}{V^2-v^2}=\frac{g}{V^2}\,dt.$

Integrating, we get

$$\frac{1}{2V}\log\frac{V+v}{V-v}=\frac{g}{V^2}t+A,\qquad \because \int\frac{1}{a^2-x^2}\,dx=\frac{1}{2a}\log\frac{a+x}{a-x}.$$

Initially when $t=0$, $v=0$ and $\log 1=0$; $\therefore$ $A=0$.

$$\therefore\ \frac{1}{2}\log\frac{V+v}{V-v}=\frac{g}{V}t.$$

But we know that $\frac{1}{2}\log\frac{a+x}{a-x}=\tanh^{-1}\frac{x}{a}.$

$$\therefore\ \tanh^{-1}\frac{v}{V}=\frac{g}{V}t\ ;\quad \therefore\ v=V\tanh\frac{g}{V}t. \qquad ...(3)$$

$\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\frac{x-a}{x+a}\quad x>a$

Time and distance relation.

Again relation between t and x is obtained by eliminating v between (2) and (3),

i.e. $v^2=V^2\left(1-e^{-\frac{2g}{V^2}x}\right)$ and $v=V\tanh\frac{g}{V}t.$

Eliminating v, we have

$$V^2\tanh^2\frac{g}{V}t=V^2\left(1-e^{-\frac{2g}{V^2}x}\right)$$

or $$e^{-\frac{2g}{V^2}x}=1-\tanh^2\frac{g}{V}t=\text{sech}^2\frac{g}{V}t$$

or $$e^{\frac{2g}{V^2}x}=\cosh^2\frac{g}{V}t \quad\text{or}\quad e^{\frac{g}{V^2}x}=\cosh\frac{g}{V}t.$$

$$\therefore\ \frac{g}{V^2}x=\log\cosh\frac{g}{V}t \text{ or } x=\frac{V^2}{g}\log\cosh\frac{g}{V}t. \quad ...(4)$$

Hence we have the following important formulae of this article which the students should commit to memory :—

1. $\frac{d^2x}{dt^2}=g\left(1-\frac{v^2}{V^2}\right).$ **2.** $v^2=V^2\left(1-e^{-\frac{2g}{V^2}x}\right).$

3. $v=V\tanh\frac{g}{V}t.$ **4.** $x=\frac{V^2}{g}\log\cosh\frac{g}{V}t.$

§ 2. Vertically upwards motion.

A particle is projected upwards against gravity (supposed constant) in a resisting medium whose resistance varies as the square of the velocity ; to find the motion.

(Delhi Hons. 58 ; Vikram 65 ; Sagar 64, 62 ; Agra 65, 46, 45)

Let the velocity of the particle at any time t at a distance x from the point of projection be v, so that the resistance due to velocity is kv^2 per unit of mass. Hence the force of resistance on the particle of mass m is mkv^2 against the direction of motion. Also the force due to gravity is mg

acting vertically downwards, i.e. against the direction of motion. Therefore using $P=mf$ we have the equation of motion of particle as $m\dfrac{d^2x}{dt^2}=-mg-mkv^2$

or
$$\frac{d^2x}{dt^2}=-g\left(1+\frac{k}{g}v^2\right)=-g\left(1+\frac{v^2}{V^2}\right), \quad \ldots(1)$$

where V is the terminal velocity equal to $\sqrt{\left(\dfrac{g}{k}\right)}$.

Velocity and distance relation.

$$v\frac{dv}{dx}=-g\left(\frac{V^2+v^2}{V^2}\right); \quad \therefore \quad \frac{2v\,dv}{V^2+v^2}=-\frac{2g}{V^2}dx.$$

Integrating, $\log(V^2+v^2)=-\dfrac{2g}{V^2}x+A.$

When $x=0$, let $v=u$ the velocity of projection, so that

$$\log(V^2+u^2)=A.$$

$$\therefore \quad \frac{2g}{V^2}x=\log(V^2+u^2)-\log(V^2+v^2)$$

or
$$\frac{2g}{V^2}x=\log\frac{V^2+u^2}{V^2+v^2}. \quad \ldots(2)$$

Time and velocity relation.

In order to find the v and t relation, we write (1) as

$$\frac{dv}{dt}=-g\left(\frac{V^2+v^2}{V^2}\right) \text{ or } \frac{dv}{V^2+v^2}=-\frac{g}{V^2}dt.$$

Integrating, $\dfrac{1}{V}\tan^{-1}\dfrac{v}{V}=-\dfrac{g}{V^2}t+B.$

Initially, when $t=0$, $v=u$; $\therefore \quad B=\dfrac{1}{V}\tan^{-1}\dfrac{u}{V}.$

$$\therefore \quad \frac{gt}{V^2}=\frac{1}{V}\left(\tan^{-1}\frac{u}{V}-\tan^{-1}\frac{v}{V}\right).$$

$$\therefore \quad t=\frac{V}{g}\left(\tan^{-1}\frac{u}{V}-\tan^{-1}\frac{v}{V}\right). \quad \ldots(3)$$

Note. Sometimes terminal velocity is denoted by K instead of V.

Hence we have the following important relation of the Is article which the students should commit to memory :—

1. $\frac{d^2x}{dt^2}=-\frac{g}{V^2}(V^2+v^2)$.

2. $\frac{2g}{V^2}x=\log\frac{V^2+u^2}{V^2+v^2}$.

3. $t=\frac{V}{g}\left(\tan^{-1}\frac{u}{V}-\tan^{-1}\frac{v}{V}\right)$, where V is the terminal velocity and u the velocity of projection.

Note. *In all the questions the procedure of § 1 and 2 will have to be reproduced in the examination*

Exercise 6

Ex. 1. *A particle is projected upwards with a velocity u in a medium whose resistance varies as the square of the velocity. Prove that it will return to the point of projection with velocity,*

$$v=\frac{uV}{\sqrt{(u^2+V^2)}} \text{ after a time } \frac{V}{g}\left(\tan^{-1}\frac{u}{V}+\tanh^{-1}\frac{v}{V}\right),$$

where V is the terminal velocity. (Delhi Hon's 55)

Resistance$=kv^2$ and if V be the terminal velocity, then

$$0=g-kV^2 \text{ or } V^2=\frac{g}{k}.$$

Equation of motion is

$$m\frac{d^2x}{dt^2}=-mg-mkv^2$$

or $$\frac{d^2x}{dt^2}=-g\left(1+\frac{k}{g}v^2\right)=-g\left(1+\frac{v^2}{V^2}\right)$$

or $$v\frac{dv}{dx}=-\frac{g}{V^2}(V^2+v^2) \quad \ldots(1)$$

or $$\frac{2v\,dv}{V^2+v^2}=-\frac{2g}{V^2}dx.$$

Integrating, $\log (V^2+v^2)=-\frac{2g}{V^2}x+A$.

When $x=0$, $v=u$; $\therefore$ $A=\log (V^2+u^2)$.

$\therefore$ $\frac{2g}{V^2}x=\log\frac{V^2+u^2}{V^2+v^2}$ which is result (2) of § 2.

Now if h be the greatest height where $v=0$, then

$$\frac{2g}{V^2}h=\log\frac{V^2+u^2}{V^2} \qquad (\because v=0)$$

or
$$h=\frac{V^2}{2g}\log\frac{V^2+u^2}{V^2}. \qquad ...(2)$$

In order to find the time for the greatest height, we write (1) as

$$\frac{dv}{dt}=-\frac{g}{V^2}(V^2+v^2) \quad \text{or} \quad \frac{dv}{V^2+v^2}=-\frac{g}{V^2}dt.$$

$$\therefore \quad \frac{1}{V}\tan^{-1}\frac{v}{V}=-\frac{g}{V^2}t+B.$$

When $t=0$, $v=u$; $\therefore$ $B=\frac{1}{V}\tan^{-1}\frac{u}{V}$.

$$\therefore \quad \frac{g}{V^2}t=\frac{1}{V}\left(\tan^{-1}\frac{u}{V}-\tan^{-1}\frac{v}{V}\right).$$

$\therefore$ $t=\frac{V}{g}\left(\tan^{-1}\frac{u}{V}-\tan^{-1}\frac{v}{V}\right)$ which is result (3) of § 2.

Putting $v=0$, we get the corresponding time t_1 for greatest height h as
$$t_1=\frac{V}{g}\tan^{-1}\frac{u}{V}. \qquad ...(3)$$

(Burdwan Hons. 65, Cal. Hon's. 61)

Note. Both the above results (2) and (3) could be directly obtained by putting $v=0$ in results (2) and (3) § 2.

Now the particle will start falling downwards and as in § 1, we have the equation of motion as

$$\frac{d^2x}{dt^2}=g-kv^2=g\left(1-\frac{v^2}{V^2}\right)=\frac{g}{V^2}(V^2-v^2).$$

or $$v\frac{dv}{dx}=\frac{g}{V^2}(V^2-v^2)$$

or $$\int_0^v \frac{2v\,dv}{V^2-v^2}=\int_0^h \frac{2g}{V^2}\,dx.$$

The limits of integration are chosen as the particle starts from rest where $x=0$ (x being measured from top) and v is the velocity when it reaches the ground after falling a distance h given by (2).

$$\therefore\quad \Big[-\log(V^2-v^2)\Big]_0^v=\frac{2g}{V^2}h$$

or $$-[\log(V^2-v^2)-\log V^2]=\frac{2g}{V^2}\cdot\frac{V^2}{2g}\log\frac{V^2+u^2}{V^2}\text{ by (2)}$$

$$\log\frac{V^2}{V^2-v^2}=\log\frac{V^2+u^2}{V^2}.$$

$$\therefore\quad \frac{V^2}{V^2-v^2}=\frac{V^2+u^2}{V^2}$$

or $$V^4=V^4+u^2V^2-v^2(V^2+u^2).$$

$$\therefore\quad v^2=\frac{u^2V^2}{V^2+u^2}\quad\text{or}\quad v=\frac{uV}{\sqrt{(u^2+V^2)}}.\qquad\text{...(5)}$$

Again we can write (4) as

$$\frac{dv}{dt}=\frac{g}{V^2}(V^2-v^2).$$

$$\therefore\quad \int_0^v\frac{dv}{V^2-v^2}=\int^{t_2}\frac{g}{V^2}\,dt.$$

t_2 corresponds to the time which the particle takes in returning to ground where its velocity is v after starting from rest from the top with zero velocity.

$$\therefore\quad \frac{1}{2V}\Big[\log\frac{V+v}{V-v}\Big]_0^v=\frac{g}{V^2}t_2.$$

$$\therefore\quad t_2=\frac{V}{g}\cdot\frac{1}{2}\log\frac{V+v}{V-v}-\log 1\Big]=\frac{V}{g}\tanh^{-1}\frac{v}{V}.\qquad\text{...(6)}$$

Hence the total time by (3) and (6), is t_1+t_2

$$=\frac{V}{g}\left[\tan^{-1}\frac{u}{V}+\tanh^{-1}\frac{v}{V}\right]. \quad ...(7)$$

Proved.

Note. You must have seen that the above question is just a reproduction of both the articles No. 1 and 2 and the answers could be easily obtained from the results of these articles by putting $v=0$ etc. But the students should reproduce them in the examination as I have done in the question.

Ex. 2. *A heavy particle is projected with velocity u in a medium the resistance of which is $gu^{-2}\tan^2\alpha$ times the square of the velocity, α being constant. Show that the particle will return to the point of projection with velocity u cos α, after time* $\frac{u}{g}\cot\alpha\left(\alpha+\log\frac{\cos\alpha}{1-\sin\alpha}\right)$. (Agra 65, 56)

It is exactly a similar type of question as Ex. 1.

Let the resistance be $kv^2=gu^{-2}\tan^2\alpha \,.\, v^2$ given.

The equation of motion is $m\frac{d^2x}{dt^2}=-mg-kv^2$

or $$v\frac{dv}{dx}=-g\left(1+\frac{v^2}{V^2}\right)$$

where $V^2=\frac{g}{k}$ is the terminal velocity.

But here $k=\frac{g}{u^2}\tan^2\alpha=\frac{g}{V^2}$; $\therefore$ $V=u\cot\alpha$. ...(1)

Now proceed with the equation $v\frac{dv}{dx}=-g\left(1+\frac{v^2}{V^2}\right)$

where $V=u\cot\alpha$ and u is the velocity of projection.

Hence from Ex. 1 result 5 the particle will return to the point of projection with velocity

$$v=\frac{uV}{\sqrt{(u^2+V^2)}}=\frac{u\,.\,u\cot\alpha}{\sqrt{(u^2+u^2\cot^2\alpha)}}=\frac{u\cos\alpha}{\sin\alpha\,.\,\text{cosec}\,\alpha}=u\cos\alpha.$$

Also from result (7) of Ex. 1 the total time taken to return to the point of projection is

$$\frac{V}{g}\left[\tan^{-1}\frac{u}{V}+\tanh^{-1}\frac{v}{V}\right].$$

Put $V=u\cot\alpha$ and $v=u\cos\alpha$.

$\therefore$ time is $\dfrac{u\cot\alpha}{g}\left[\tan^{-1}\dfrac{u}{u\cot\alpha}+\tanh^{-1}\dfrac{u\cos\alpha}{u\cot\alpha}\right]$

$=ug^{-1}\cot\alpha\,[\tan^{-1}\tan\alpha+\tanh^{-1}\sin\alpha]$. ...(2)

But $\tan^{-1}x=\dfrac{1}{2}\log\dfrac{1+x}{1-x}$.

$\therefore$ $\tanh^{-1}\sin\alpha=\dfrac{1}{2}\log\dfrac{1+\sin\alpha}{1-\sin\alpha}$.

$=\dfrac{1}{2}\log\dfrac{(1+\sin\alpha)(1-\sin\alpha)}{(1-\sin\alpha)^2}$.

$=\dfrac{1}{2}\log\dfrac{\cos^2\alpha}{(1-\sin\alpha)^2}=\frac{1}{2}.2\log\dfrac{\cos\alpha}{1-\sin\alpha}=\log\dfrac{\cos\alpha}{1-\sin\alpha}$.

$\therefore$ time is $ug^{-1}\cot\alpha\left[\alpha+\log\dfrac{\cos\alpha}{1-\sin\alpha}\right]$ from (2)

Ex. 3. *Two particles move in a medium whose resistance varies as square of the velocity. One is let fall from a height h and the other projected upwards at the same instant with initial velocity sufficient to carry it to a height h. Show that the particles meet at a depth y below the highest point given by* $\cosh\beta\cos(\alpha-\beta)=1$ *where* $gy=V^2\log\cosh\beta$ *and* $gh=V^2\sec\alpha$, *V being the terminal velocity.*



Suppose the particles meet at P after time t at the depth y below O; then from result (4) § 1, we have

$$y=\frac{V^2}{g}\log\cosh\frac{g}{V}t.$$

$\therefore$ $gy=V^2\log\cosh\beta$ where $\beta=\dfrac{g}{V}t$ ...(1)

Again if a particle be projected upwards with velocity u, then

$$\frac{2g}{V^2}x=\log\frac{V^2+u^2}{V^2+v^2} \text{ from (2) of § 2.} \quad ...(2)$$

If h be the greatest height, then $v=0$ when $x=h$.

$$\therefore \frac{2g}{V^2}h=\log\frac{V^2+u^2}{V^2}=\log\left(1+\frac{u^2}{V^2}\right). \quad ...(3)$$

If we put $\frac{u}{V}=\tan\alpha$, then

$$\frac{2g}{V^2}h=\log(1+\tan^2\alpha)=\log\sec^2\alpha=2\log\sec\alpha.$$

$$\therefore \quad gh=V^2\log\sec\alpha. \quad ...(4)$$

Now if t be time when the velocity is v at P where $x=OP=h-y$, then [*mind this t is same as in (1) as the* particle is projected at the same time when the other falls]

$$t=\frac{V}{g}\left(\tan^{-1}\frac{u}{V}-\tan^{-1}\frac{v}{V}\right) \text{ from (3) of § 2,} \quad ...(5)$$

and $$\frac{2g}{V^2}(h-y)=\log\frac{V^2+u^2}{V^2+v^2} \text{ by (2) of above.}$$

$$\therefore \quad \frac{2gh}{V^2}-\frac{2g}{V^2}y=\log\left(1+\frac{u^2}{V^2}\right)-\log\left(1+\frac{v^2}{V^2}\right)$$

or $$\frac{2g}{V^2}y=\log\left(1+\frac{v^2}{V^2}\right) \text{ by (3)}$$

or $$2\log\cosh\beta=\log\left(1+\frac{u^2}{V^2}\right) \text{ by (1)}$$

or $$\cosh^2\beta=1+\frac{v^2}{V^2} \quad \text{or} \quad \cosh^2\beta-1=\frac{v^2}{V^2}$$

or $$\sinh\beta=\frac{v}{V}. \quad ...(6)$$

Now $$\frac{gt}{V}=\left(\tan^{-1}\frac{u}{V}-\tan^{-1}\frac{v}{V}\right) \text{ by (5)}$$

or $$\beta=\tan^{-1}\tan\alpha-\tan^{-1}\sinh\beta \text{ by (1), (6)}$$

or $$\tan^{-1}\sinh\beta=(\alpha-\beta), \quad \therefore \quad \sinh\beta=\tan(\alpha-\beta).$$

$$\therefore \quad \cos(\alpha-\beta)=\frac{1}{\sqrt{\{1+\tan^2(\alpha-\beta)\}}}=\frac{1}{\sqrt{(1+\sinh^2\beta)}}=\frac{1}{\cosh\beta}$$

$$\therefore \quad \cosh\beta\cos(\alpha-\beta)=1. \qquad \ldots(7)$$

Results (1), (4) and (7) prove the results.

Ex. 4. *A heavy particle is projected vertically upwards in a medium the resistance of which varies as square of the velocity. It has a kinetic energy K in its upward path at a given point. When it passes the same point on the way down, show that its loss of energy is $\frac{K^2}{K+K'}$ where K' is the limit to which the energy approaches in its downward course.*

(Delhi Hons. 55, 58 ; Agra 53)

Let the resistance be kv^2 and V be the terminal velocity so that $\qquad g-kV^2=0$ or $V^2=g/k$.

Hence K' the limit to which the energy approaches in downward course is $\frac{1}{2}mV^2$ i.e. $K'=\frac{1}{2}mV^2$. $\qquad \ldots(1)$

Since any point may be taken as the given point we take it the point of projection where the velocity of projection is u, so that $\qquad K=\frac{1}{2}mu^2$. $\qquad \ldots(2)$

Now proceeding as in § 2 or in Ex. 1 if h be the greatest height atttained by the particle, we have from (2) of Ex. 1 p. 248

$$h=\frac{V^2}{2g}\log\frac{V^2+u^2}{V^2}. \qquad \ldots(3)$$

Again if u be the velocity when the particle returns to the same point after falling from rest from its greatest height h, then from result (5) of Ex. 1 p. 249, we have

$$v=\frac{uV}{\sqrt{(u^2+V^2)}}.$$

If K_1 be the energy when it passes through the same point, then $\qquad K_1=\frac{1}{2}mv^2$

or $\qquad K=\frac{1}{2}m\frac{u^2V^2}{u^2+V^2}.$

$\therefore$ loss of energy is $K-K_1=\frac{1}{2}mu^2-\frac{1}{2}m\,\frac{u^2V^2}{u^2+V^2}$

or $\quad \text{loss}=\frac{1}{2}mu^2\left[1-\frac{V^2}{u^2+V^2}\right]=K.\frac{u^2}{u^2+V^2}$

or $\quad \text{loss}=K.\frac{\frac{1}{2}mu^2}{\frac{1}{2}mu^2+\frac{1}{2}mV^2}=K\,\frac{K}{K+K'}=\frac{K^2}{K+K'}.$

Ex. 5. *A particle falls from rest under gravity in a medium whose resistance varies as the square of the velocity If v be its velocity and v_0 the velocity which would be acquired if there was no resistance and V the terminal velocity, show*

that $$\frac{v^2}{v_0^2}=1-\frac{1}{2}\,\frac{v_0^2}{V^2}+\frac{1}{2.3}\,\frac{v_0^4}{V^4}-\ldots$$

Proceeding exactly as in § 1, we have the velocity at any distance x is given by the relation (2) P. 244

i. e. $$v^2=V^2\,(1-e^{-2gx/V^2}) \qquad \ldots(1)$$

Again if there were no medium and v_0 be the corresponding velocity after falling a distance x, then

$$v_0^2=2gx. \qquad \ldots(2)$$

$\therefore$ Eliminating x between (1) and (2), we have

$$v^2=V^2\,[1-e^{-v_0^2/V^2}]$$

or $$v^2=V^2\left[1-\left\{1-\frac{v_0^2}{V^2}+\frac{1}{2\,!}\frac{v_0^4}{V^4}-\frac{1}{3\,!}\frac{v_0^6}{V^6}+\ldots\right\}\right]$$

or $$v^2=v_0^2-\frac{1}{2}\,\frac{v_0^4}{V^2}+\frac{1}{2.3}\,\frac{v_0^6}{V^4}-\ldots$$

or $$\frac{v^2}{v_0^2}=1-\frac{1}{2}\,\frac{v_0^2}{V^2}+\frac{1}{2.3}\,\frac{v_0^4}{V^4}-\ldots$$ Proved.

Ex. 6. *A particle moves from rest at a distance a from a fixed point O under the action of a force equal to μ times the distance per unit of mass. If the resistance of the medium in which it moves be k times the square of the velocity per*

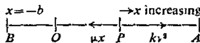
unit mass, show that square of its velocity when it is at a distance x from O is

$$\frac{\mu x}{k}-\frac{\mu a}{k}\ e^{2k\,(x-a)}+\frac{\mu}{2k^2}\ [1-e^{2k\,(x-a)}].$$

Show also that when it first comes to rest it will be at a distance b given by $(1-2bk)\ e^{2bk}=(1+2ak)\ e^{-2ak}$.

(Cal. Hon's. 62)

Equation of motion is



$$m\frac{d^2x}{dt^2}=-m\mu x+mkv^2.$$

Force in the sense of x decreasing −ive and those in the sense of x increasing +ive

or $$v\frac{dv}{dx}=-\mu x+kv^2$$

or $$\tfrac{1}{2}\frac{dv^2}{dx}-kv^2=-\mu x$$

or $$\frac{dv^2}{dx}-2kv^2=-2\mu x.$$

Above is a linear differential equation and I. F.$=e^{-2kx}$.
Multiplying both sides by I F. and integrating, we get

$$v^2e^{-2kx}=-2\mu\int xe^{-2kx}\,dx+C.\quad \text{Integrate by parts}$$

$$=-2\mu\left[x\,\frac{(e^{-2kx})}{-2k}-(1)\,\frac{e^{-2kx}}{4k^2}\,dx\right]+C$$

or $$v^2e^{-2kx}=\frac{\mu}{k}\,x.e^{-2kx}+\frac{\mu}{2k^2}\,e^{-2kx}+C$$

or $$v^2=\frac{\mu}{k}\,x+\frac{\mu}{2k^2}+Ce^{2kx}.\qquad ...(1)$$

In order to find C we know that when $x=a$, $v=0$.

$$\therefore\quad 0=\frac{\mu a}{k}+\frac{\mu}{2k^2}+Ce^{2ka}.$$

$$\therefore \quad C=-\left(\frac{\mu a}{k}+\frac{\mu}{2k^2}\right)e^{-2ka}.$$

Putting for C in (1), we get

$$v^2=\frac{\mu x}{k}+\frac{\mu}{2k^2}-\left(\frac{\mu a}{k}+\frac{\mu}{2k^2}\right)e^{-2ka}.e^{2kx}$$

or
$$v^2=\frac{\mu x}{k}+\frac{\mu}{2k^2}(1-e^{2k(x-a)})-\frac{\mu a}{k}e^{2k(x-a)}. \quad ...(2)$$

The velocity V at the origin is obtained by putting $x=0$ in (2), and is
$$V^2=\frac{\mu}{2k^2}(1-e^{-2ka})-\frac{\mu a}{k}e^{-2ka}$$

$$=\frac{\mu}{2k^2}-\frac{\mu}{k}e^{-2ka}\left(a+\frac{1}{2k}\right). \quad ...(3)$$

We observe that the particle has got some velocity at the origin and let us suppose that it comes to rest at a distance b from origin at B where $x=-b$. Putting $v=0$ and $x=-b$ in (2), we get

$$0=\frac{\mu}{k}(-b)+\frac{\mu}{2k^2}(1-e^{-2k(b+a)})-\frac{\mu a}{k}e^{-2k(b+a)}.$$

Multiplying by $2k^2$, we get

$$(1+2ak)e^{-2kb}.e^{-2ka}=(1-2bk)$$

or
$$(1+2ak)e^{-2ka}=(1-2bk)e^{2kb}. \qquad \text{Proved.}$$

Ex. 7. *A particle falls from rest at a distance a from the centre of the earth, the motion meeting a small resistance proportional to the square of the velocity v and the retardation being μ for unit velocity. Show that the kinetic energy at a distance x from the centre is*

$$mgr^2\left[\frac{1}{x}-\frac{1}{a}+2\mu\left(1-\frac{x}{a}\right)-2\mu\log\frac{a}{x}\right]$$

the square of μ being neglected and r being the radius of the earth.

We know that the centre of the earth attracts the particle towards it with an acceleration varying inversely as the square of the distance and its value when the particle is at a distance x from O is $\frac{\lambda}{x^2}$. But at the surface of the earth, where $x=r$ the radius of the earth acceleration is g. or $\frac{\lambda}{r^2}=g$ or $\lambda=gr^2$. Hence the attraction of the earth at a distance x is $\frac{gr^2}{x^2}$. If v be the velocity at a distance x, then resistance is μv^2. Taking the forces in the sense of x increasing +ive and in the sense of x decreasing −ive, we have the equation of motion as

$$v\frac{dv}{dx}=\mu v^2-\frac{gr^2}{x^2} \quad \text{or} \quad \tfrac{1}{2}\frac{dv^2}{dx}-\mu v^2=-\frac{gr^2}{x^2}$$

or

$$\frac{dv^2}{dx}-2\mu v^2=-2g\frac{r^2}{x^2}.$$

Above is a linear differential equation and I. F. is $e^{-2\mu x}$ Multiplying both sides by I. F. and integrating, we get

$$v^2e^{-2\mu x}=-2gr^2\int e^{-2\mu x}\frac{1}{x^2}\,dx+C,$$

But $e^{-2\mu x}=1-2\mu x+\ldots$ neglecting μ^2 and higher powers.

$$\therefore \quad v^2(1-2\mu x)=-2gr^2\int\frac{1-2\mu x}{x^2}\,dx+C,$$

$$v^2(1-2\mu x)=-2gr^2\int\left(\frac{1}{x^2}-\frac{2\mu}{x}\right)dx+C,$$

$$v^2(1-2\mu x)=-2gr^2\left[-\frac{1}{x}-2\mu\log x\right]+C.$$

Initially when $x=a$, $v=0$; $\therefore \quad C=-2gr^2\left[\frac{1}{a}+2\mu\log a\right].$

$$\therefore \quad v^2(1-2\mu x)=2gr^2\left[\frac{1}{x}-\frac{1}{a}+2\mu\,(\log x-\log a\right]$$

or $$v^2=2gr^2(1-2\mu x)^{-1}\left[\frac{1}{x}-\frac{1}{a}+2\mu\log\frac{x}{a}\right]$$

or $$v^2=2gr^2(1+2\mu x+\dots)\left[\frac{1}{x}-\frac{1}{a}+2\mu\log\frac{x}{a}\right]$$

$$=2gr^2\left[\frac{1}{x}-\frac{1}{a}+2\mu x\left(\frac{1}{x}-\frac{1}{a}\right)+2\mu\log\frac{x}{a}\right],$$

we have neglected the term of μ^2 etc.

$$\therefore \quad \text{K.E.}=\frac{1}{2}mv^2=mgr^2\left[\frac{1}{x}-\frac{1}{a}+2\mu\left(1-\frac{x}{a}\right)-2\mu\log\frac{a}{x}\right].$$

Ex. 8. *An attracting force varying as the distance acts on a particle initially at rest at a distance a Show that if V be the velocity when the particle is at a distance x and V′ the velocity when the resistance of air is taken into account, then* $V'=V\left[1-\frac{1}{3}k\frac{(2a+x)(a-x)}{a+x}\right]$ *nearly, the resistance being k times the square of the velocity, k being very small.*

In the first case when there is no resistance, then

$$v\frac{dv}{dx}=-\mu x$$

O x P A

$$\therefore \quad v^2=-\mu x^2+A.$$

When $x=a$, $v=0$; $\therefore$ $A=\mu a^2$.

$$\therefore \quad v^2=\mu(a^2-x^2) \quad \text{or} \quad V^2=\mu(a^2-x^2). \qquad \dots(1)$$

When the resistance of air kv^2 is taken into account, then it will act in the sense of x increasing

$$\therefore \quad v\frac{dv}{dx}=-\mu x+kv^2 \quad \text{or} \quad \frac{dv^2}{dx}-2kv^2=-2\mu x.$$

Above is linear and I. F. is e^{-2kx}. Multiplying by I. F. and integrating, we get

$$v^2 . e^{-2kx}=-2\mu\int xe^{-2kx}\,dx+C.$$

or $v^2(1-2kx+\ldots)=-2\mu\int x(1-2kx+\ldots)\,dx+C$

k being small, its squares and higher powers are neglected.

or $$v^2(1-2kx)=-2\mu\left[\frac{x^2}{2}-\frac{2kx^3}{3}\right]dx+C.$$

Initially when $x=a$, $v=0$. $\therefore$ $C=2\mu\left[\frac{a^2}{2}-\frac{2ka^3}{3}\right]$

$$\therefore\ v^2(1-2kx)=2\mu.\left[\frac{a^2-x^2}{2}-\frac{2k}{3}(a^3-x^3)\right]$$

or $$v^2=2\mu(1-2kx)^{-1}\left[\frac{a^2-x^2}{2}-\frac{2k}{3}(a^3-x^3)\right]$$

or $$v^2=2\mu(1+2kx-\ldots)\left[\frac{a^2-x^2}{2}-\frac{2k}{3}(a^3-x^3)\right]$$

or $$v^2=2\mu\left[\frac{a^2-x^2}{2}+kx\ (a^2-x^2)-\frac{2k}{3}(a^3-x^3)\right]$$

neglecting k^2 etc.

or $$v^2=2\mu\,\frac{a^2-x^2}{2}\left[1+2kx-\frac{2k}{3}\cdot\frac{2(a^3-x^3)}{(a^2-x^2)}\right].$$

Since the velocity is given to be V' and $V^2=\mu(a^2-x^2)$ by (1).

$$\therefore\ V'^2=V^2\left[1+2kx-\frac{4k}{3}\frac{a^2+x^2+ax}{a+x}\right]$$

or $$V'^2=V^2\left[1+\frac{2k}{3}\cdot\frac{(3xa+3x^2-2a^2-2x^2-2ax)}{a+x}\right]$$

or $$V'=V\left[1-\frac{2k}{3}\frac{(2a+x)(a-x)}{a+x}\right]^{1/2}$$

or $$V'=V\left[1-\frac{1}{2}\cdot\frac{2k}{3}\frac{(2a+x)(a-x)}{a+x}+\ldots\right]$$

neglecting k^2 etc.

or $$V'=V\left[1-\frac{k}{3}\frac{(2a+x)(a-x)}{a+x}\right].$$

Proved.

Ex. 9. *A particle moving in a straight line is subject to a resistance kv^3 where v is the velocity. Show that if v is the velocity at time t when the distance is s,*

$$v=\frac{u}{1+kus},\ t=\frac{s}{u}+\frac{1}{2}ks^2,$$

where u is the initial velocity.

The equation of motion is

$$v\frac{dv}{ds}=-kv^3 \quad \text{or} \quad -\frac{1}{v^2}\,dv=k\,ds.$$

Integrating, $\frac{1}{v}=ks+A$. When $v=u$, $s=0$; $\therefore\ A=\frac{1}{u}$.

$$\therefore\ \frac{1}{v}=ks+\frac{1}{u}=\frac{1+ksu}{u} \quad \text{or} \quad v=\frac{u}{1+ksu}. \quad ...(1)$$

Now $\frac{ds}{dt}=v=\frac{u}{1+ksu}$; $\therefore\ \frac{1}{u}(1+ksu)\,ds=dt$.

Integrating, $\frac{1}{u^2k}\frac{(1+ksu)^2}{2}=t+B$.

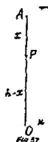
When $t=0$, $s=0$. $\therefore\ B=\frac{1}{2u^2k}$.

$$\therefore\ t=\frac{1}{2u^2k}[(1+ksu)^2-1]=\frac{1}{2u^2k}[2ksu+k^2s^2u^2]$$

or

$$t=\frac{s}{u}+\frac{1}{2}ks^2. \quad ...(2)$$

Ex. 10. *If the resistance vary as the fourth power of the velocity the energy of m lbs. at a depth x below the highest point when moving in a vertical line under gravity will be $E\tan\left(\frac{mgx}{E}\right)$ when rising and $E\tanh\left(\frac{mgx}{E}\right)$ when falling where E is the terminal energy in the medium.*



When the particle is falling, the equation of motion is

$$m\frac{d^2x}{dt^2}=mg-mkv^4$$

or $$\frac{d^2x}{dt^2}=g-kv^4=g\left(1-\frac{k}{g}v^4\right).$$

If V be the terminal velocity, then acceleration is zero.

$$\therefore\quad 0=g\left(1-\frac{k}{g}V^4\right) \text{ or } \frac{k}{g}=\frac{1}{V^4}.$$

$$\therefore\quad v\frac{dv}{dx}=g\left(1-\frac{v^4}{V^4}\right)=\frac{g}{V^4}(V^4-v^4).$$

$$\therefore\quad \frac{2v\,dv}{(V^2)^2-(v^2)^2}=\frac{2g}{V^4}\,dx.$$

Integrating, we get $\dfrac{1}{2V^2}\log\dfrac{V^2+v^2}{V^2-v^2}=\dfrac{2g}{V^4}x+A.$

When $x=0$, $v=0$, $\therefore\ A=0$. $\because\ \log 1=0$.

$$\therefore\quad \frac{1}{2V^2}\log\frac{V^2+v^2}{V^2-v^2}=\frac{2g}{V^4}x$$

or $$\tanh^{-1}\frac{v^2}{V^2}=\frac{2g}{V^2}x,\quad \because\ \frac{1}{2}\log\frac{a+x}{a-x}=\tanh^{-1}\frac{x}{a}$$

or $$v^2=V^2\tanh\frac{2g}{V^2}x.$$

$$\therefore\quad \text{energy}=\frac{1}{2}mv^2=\frac{1}{2}mV^2\tanh\frac{2g}{V^2}x. \qquad \ldots(1)$$

But $E=\text{terminal energy}=\dfrac{1}{2}mV^2$ or $V^2=\dfrac{2E}{m}$. ...(2)

$\therefore$ energy at a depth x below the highest point while falling is $E\tanh\dfrac{mgx}{E}$, by (1) and (2) which proves the second part.

Motion vertically up.

In this case $m\dfrac{d^2y}{dt^2}=-mg-mkv^4$

or $$\frac{d^2y}{dt^2}=-g\left(1+\frac{k}{g}v^4\right)=-g\left(1+\frac{v^4}{V^4}\right).$$

or $$v\frac{dv}{dy}=-\frac{g}{V^4}(V^4+v^4) \quad \text{or} \quad \frac{2v\,dv}{(V^2)^2+(v^2)^2}=-\frac{2g}{V^4}\,dy.$$

or $$\frac{1}{V^2}\tan^{-1}\frac{v^2}{V^2}=-\frac{2g}{V^4}\,y+C.$$

Let initially the velocity of projection be u, so that when $y=0$, $v=u$.

$$\therefore \quad C=\frac{1}{V^2}\tan^{-1}\frac{u^2}{V^2};$$

$$\therefore \quad \frac{2gy}{V^2}=\left(\tan^{-1}\frac{u^2}{V^2}-\tan^{-1}\frac{v^2}{V^2}\right). \qquad \text{...(3)}$$

Hence if v be the velocity at P where $y=h-x$, then

$$\frac{2g}{V^2}(h-x)=\tan^{-1}\frac{u^2}{V^2}-\tan^{-1}\frac{v^2}{V^2}. \qquad \text{...(4)}$$

But we are given that the greatest height is h and hence from (3), when $y=h$, $v=0$; $\therefore \quad \dfrac{2gh}{V^2}=\tan^{-1}\dfrac{u^2}{V^2}.$ ...(5)

Hence subtracting (4) and (5), we get

$$-\frac{2g}{V^2}x=-\tan^{-1}\frac{v^2}{V^2}; \quad \therefore \quad v^2=V^2\tan\frac{2g}{V^2}x.$$

$$\therefore \quad \text{energy}=\frac{1}{2}mv^2=\frac{1}{2}mV^2\tan\frac{2g}{V^2}x. \qquad \text{...(6)}$$

But $E=\text{terminal energy}=\dfrac{1}{2}mV^2$ or $V^2=\dfrac{2E}{M}$. ...(7)

$\therefore$ by (6) and (7) energy at depth x below the highest point while the particle is rising is $E\tan\left(\dfrac{mgx}{E}\right)$, which proves the 1st part.

Ex. 11. *A particle is projected with velocity V along a smooth horizontal plane in a medium whose resistance per unit of mass is μ times the cube of the velocity. Show that the distance it has described in time t is*

$$\frac{1}{\mu V}[\sqrt{(1+2\mu V^2 t)}-1]$$

and that its velocity then is $\frac{V}{\sqrt{(1+2\mu V^2 t)}}$.

Equation of motion is

$$m\frac{d^2x}{dt^2}=-m\mu v^3 \quad \text{or} \quad \frac{d^2x}{dt^2}=-\mu v^3. \qquad ...(1)$$

Here we are to find the relation between x, t and v, t.

We write (1) as $\frac{dv}{dt}=-\mu v^3$ or $\frac{dv}{-v^3}=\mu\, dt$

Integrating, $\frac{1}{2v^2}=\mu t+C$ when $t=0$, $v=V$; $\therefore$ $c=\frac{1}{2V^2}$.

$$\therefore \quad \frac{1}{2v^2}=\mu t+\frac{1}{2V^2}=\frac{1+2V^2\mu t}{2V^2}.$$

$$\therefore \quad v=\frac{V}{\sqrt{(1+2V^2\mu t)}}. \qquad ...(2)$$

Hence proved.

Again $\quad v=\frac{dx}{dt}=\frac{V}{\sqrt{(1+2V^2\mu t)}}$.

Integrating,

$$x=V\cdot\frac{1}{2V^2\mu}\cdot 2\sqrt{(1+2V^2\mu t)}+B. \quad \because \int\frac{1}{\sqrt{x}}\,dx=2\sqrt{x}.$$

Initially when $t=0$, $x=0$. $\quad\therefore\quad B=-\frac{1}{\mu V}$.

$$\therefore \quad x=\frac{1}{\mu V}[\sqrt{(1+2V^2\mu t)}-1].$$

Ex. 12. *A particle of mass m is projected vertically under gravity, the resistance of the air being mk times the velocity.*

.t height attained by the particle is

$$\frac{V^2}{g}\,[\lambda-\log(1+\lambda)],$$

.e terminal velocity of the particle and λV is the .ical velocity. (Sagar 64 ; Agra 66)

.t the particle falls from rest, then $\frac{d^2x}{dt^2}=g-kv$. If V be the velocity when acceleration is zero, then $0=g-kV$ or $V=\frac{g}{k}$. This velocity V is called the terminal velocity.

Now the particle is projected up ; hence the equation of motion is

$$m\frac{d^2x}{dt^2}=-mg-mkv$$

or

$$v\frac{dv}{dx}=-g\left(1+\frac{k}{g}v\right)=-g\left(1+\frac{v}{V}\right)$$

or

$$v\frac{dv}{dx}=-\frac{g}{V}(V+v)$$

or

$$\frac{v}{V+v}\,dv=-\frac{g}{V}\,dx$$

or

$$\left(1-\frac{V}{V+v}\right)dv=-\frac{g}{V}\,dx.$$

Integrating, we get $v-V\log(V+v)=-\frac{g}{V}x+C.$

Initially when $x=0$, $v=\lambda V$.

$\therefore\ \lambda V-V\log(V+\lambda V)=C.$

$\therefore\ v-V\log(V+v)-\lambda V+V\log(V+\lambda V)=-\frac{g}{V}x.$

If h be the greatest height, then at $x=h$, $v=0$.

$\therefore\ -V\log V-\lambda V+V\log(V+\lambda V)=-\frac{g}{V}h$

or

$$h=\frac{V}{g}\left(\lambda V-V\log\frac{V+\lambda V}{V}\right)$$

Ex. 13. *A particle, of mass m, is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, show that the distance fallen through in time t is*

$$g\,\frac{m^2}{\mu^2}\left[e^{-\mu t/m}-1+\frac{\mu t}{m}\right].\qquad\text{(Sagar 62)}$$

Total resistance $=\mu v$.

Equation of motion is $m\,\dfrac{d^2x}{dt^2}=mg-\mu v$.

$$\therefore\quad \frac{dv}{dt}=g-\frac{\mu}{m}.v\quad\text{or}\quad \frac{dv}{dt}+\frac{\mu}{m}\,v=g.$$

Above is linear and I. F. is $e^{\mu t/m}$. Multiplying both sides by I. F. and integrating, we get

$$v.e^{\mu t/m}=\int g e^{\mu t/m}\,dt+C=g.\frac{m}{\mu}\,e^{\mu t/m}+C.$$

Initially when $t=0$, $v=0$; $\therefore\quad C=-g\,\dfrac{m}{\mu}$.

$$\therefore\quad ve^{\mu t/m}=\frac{mg}{\mu}\,[e^{\mu t/m}-1].$$

$$v=\frac{mg}{\mu}\,[1-e^{-\mu t/m}]$$

or

$$\frac{dx}{dt}=\frac{mg}{\mu}\,[1-e^{-\mu t/m}].$$

Integrating again, we get

$$x=\frac{mg}{\mu}\left[t+\frac{m}{\mu}\,e^{-\mu t/m}\right]+B.$$

Initially when $t=0$, $x=0$; $\therefore\quad B=-\dfrac{m^2g}{\mu^2}$.

$$\therefore\quad x=\frac{mg}{\mu}\left[t+\frac{m}{\mu}\,e^{-\mu t/m}\right]-\frac{m^2g}{\mu^2}$$

or $$t=\frac{m^2g}{\mu^2}\left[\frac{\mu t}{m}+e^{-\mu t/m}-1\right].$$ Proved.

Ex. 14. *A heavy particle is projected in a resisting medium, the resistance varying as the velocity. If v_1 and v_2 are its velocities at any point in its upward and downward path and t be the interval between its passage through this point, prove that $v_1+v_2=gt$, $V-v_2=(V+v_1)\,e^{-gt/V}$, where V is the terminal velocity*

Resistance is kv.

For a particle falling, we have

$$\frac{d^2x}{dt^2}=g-kv=g\left(1-\frac{kv}{g}\right).$$



Acceleration is zero if $v=V$, so that $V=\frac{g}{k}$.

$$\therefore\quad \frac{d^2x}{dt^2}=g\left(1-\frac{v}{V}\right),$$

where V is called the terminal velocity.

Now any point may be taken as the point of projection. So we consider a point P on its upwards path as the point of projection where its velocity is v_1 given. Let t_1 be the time to reach its greatest height h above P and then it arrives at O from where it falls and returns to P again with velocity say v_2 and time taken for the return journey be t_2, so that total time is t.

$$\therefore\quad t=t_1+t_2.$$

Equation of motion for upward motion.

$$\frac{d^2x}{dt^2}=-g-kv=-g\left(1+\frac{v}{V}\right)$$

or $$v\frac{dv}{dx}=-\frac{g}{V}(V+v). \qquad \ldots(1)$$

$$\frac{v+V-V}{V+v}\,dv=-\frac{g}{V}\,dx$$

or $$\int\left(1-\frac{V}{V+v}\right)dv=\int -\frac{g}{V}\,dx.$$

Integrating, $v-V\log(V+v)=-\frac{g}{V}x+A.$

When $x=0$, $v=v_1$ given. $\therefore\ v_1-V\log(V+v_1)=A.$

$$\therefore\ v-V\log(V+v)=-\frac{g}{V}x+v_1-V\log(V+v_1).$$

$$\therefore\ \frac{g}{V}x=v_1-v+V\log\frac{V+v}{V+v_1}.$$

At the greatest height $x=h$, $v=0$.

$$\therefore\ \frac{g}{V}h=v_1+V\log\frac{V}{V+v_1}$$

or $$h=\frac{Vv_1}{g}+\frac{V^2}{g}\log\frac{V}{V+v_1}. \quad \text{...(2)}$$

Again in order to find time for greatest height we write the equation of motion (1) as

$$\frac{dv}{dt}=-\frac{g}{V}(V+v) \quad \text{or} \quad \frac{dv}{V+v}=-\frac{g}{V}\,dt$$

Integrating, $\log(V+v)=-\frac{g}{V}t+B.$

When $t=0$, $v=v_1$, $\therefore\ B=\log(V+v_1).$

$$\therefore\ \log\frac{(V+v)}{V+v_1}=-\frac{g}{V}t.$$

At the highest point $v=0$ and if time taken be t_1, then

$$\log\frac{V}{V+v_1}=-\frac{g}{V}t_1, \quad \therefore\ t_1=-\frac{V}{g}\log\frac{V}{V+v_1}. \quad \text{...(3)}$$

For downward motion the equation of motion is

$$v\frac{dv}{dx}=g-kv=g\left(1-\frac{v}{V}\right)=\frac{g}{V}(V-v). \quad \text{...(4)}$$

$$\therefore\ \frac{-v\,dv}{V-v}=-\frac{g}{V}\,dx \ . \ \text{or} \ \frac{V-v-V}{V-v}\,dv=-\frac{g}{V}\,dx$$

T_1=time of ascent$=\frac{u}{g}$, T_2=time of descent$=\frac{u}{g}$

and H=greatest height$=\frac{u^2}{2g}$.

Replacing the velocity of projection v_1 of Q. 14 by u, we have the corresponding data when there is resistance as following :—

$$t_1=\text{time of ascent}=-\frac{V}{g}\log\frac{V}{V+u}$$

[result 3 of Q. 14 P. 267]

or $$t_1=\frac{V}{g}\log\frac{V+u}{V}=\frac{V}{g}\log\left(1+\frac{u}{V}\right)=\frac{V}{g}\left(\frac{u}{V}-\frac{u^2}{2V^2}+\dots\right)$$

we have neglected other terms as u/V is small

or $$t_1=\frac{u}{g}-\frac{u^2}{2Vg}=T_1-T_1\cdot\frac{u}{2V}.$$

$\therefore$ decrease in time$=T_1-t_1=T_1\frac{u}{2V}=\frac{u}{2V}$ of the time when there is no resistance.

h=greatest height$=\frac{Vu}{g}+\frac{V^2}{g}\log\frac{V}{V+u}$ by writing u for v_1 in (2) of Q. 14 P. 267.

or $$h=\frac{Vu}{g}-\frac{V^2}{g}\log\left(1+\frac{u}{V}\right)$$

$$=\frac{Vu}{g}-\frac{V^2}{g}\left(\frac{u}{V}-\frac{1}{2}\frac{u^2}{V^2}+\frac{1}{3}\frac{u^3}{V^3}\right)$$

or $$h=\frac{Vu}{g}-\frac{Vu}{g}+\frac{u^2}{2g}-\frac{u^3}{3Vg}=H-\frac{2}{3}\cdot\frac{u}{V}H.$$

$\therefore$ decrease in height$=H-h=\frac{2}{3}\frac{u}{V}H=\frac{2}{3}\frac{u}{V}$ of the height when there is no resistance. For third part take the help of result 7 of Q. 14.

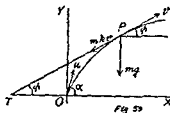
Ex. 16. *A particle is projected vertically upwards with a velocity v in a medium whose resistance is kv^4 per unit of mass, v being the velocity. Show that the greatest height reached is* $\frac{1}{2\sqrt{(kg)}}\tan^{-1}\left\{\sqrt{\left(\frac{k}{g}\right)}\,v^2\right\}$. (Osmania 66)

§ 3. **Motion of a projectile in a resisting medium in which resistance varies as velocity.**

A particle is projected under gravity and a resistance equal to mk (velocity) with a velocity u at an angle α to the horizon. To find the motion.

(Vikram 64, Sagar 63 ; Agra 51, 63, Punjab 57, 61)

Let P be the position of the particle after time t and tangent at P be inclined at an angle ψ to the axis, so that $\frac{dy}{dx}=\tan\psi$.



$\therefore \quad \sin\psi=\frac{dy}{ds}$ and $\cos\psi=\frac{dx}{ds}$.

Taking OX and OY as the axes of co-ordinates, let the co-ordinates of P be (x, y). There are two forces acting on the particle, *i.e* its weight mg vertically downwards and resistance kv along the tangent opposite to the direction of motion. We will write down the equations of motion along the axes of co-ordinates, accelerations along which are $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ respectively. Using $mf=P$, we have the following equations :

$$m\frac{d^2x}{dt^2}=-mkv\cos\psi \text{ or } \frac{d^2x}{dt^2}=-k.\frac{ds}{dt}.\frac{dx}{ds}=-k.\frac{dx}{dt}. \quad ...(1)$$

or $$\frac{\ddot{x}}{\dot{x}}=-k.$$

Integrating, we get $\log\frac{dx}{dt}=-kt+A$.

Initially when $t=0$, $\frac{dx}{dt}=$ initial horizontal component of velocity $=u\cos\alpha$.

$$\therefore \quad \log u\cos\alpha=A.$$

$$\therefore \quad \log\frac{\dot{x}}{u\cos\alpha}=-kt \text{ or } \frac{dx}{dt}=u\cos\alpha.e^{-kt}. \quad ...(2)$$

The other equation of motion is

$$m\frac{d^2y}{dt^2}=-mg-mkv\sin\psi$$

or $$\frac{d^2y}{dt^2}=-g-k\,.\,\frac{ds}{dt}\cdot\frac{dy}{ds}=-g-k\frac{dy}{dt} \qquad \ldots(3)$$

or $$\frac{k\ddot{y}}{g+k\dot{y}}=-k.$$

Integrating, we get $\log(g+k\dot{y})=-kt+B$.

Initially when $t=0$,

$\dot{y}$=initial vertical component of velocity$=u\sin\alpha$.

$$\therefore\quad \log(g+ku\sin\alpha)=B.$$

$$\therefore\quad \log\frac{g+k\dot{y}}{g+ku\sin\alpha}=-kt.$$

$$\therefore\quad g+k\frac{dy}{dt}=(g+ku\sin\alpha)\,e^{-kt}. \qquad \ldots(4)$$

Equations (2) and (4) give the horizontal and vertical components of velocity at any time.

Again integrating equation (2), *i.e.* $\frac{dx}{dt}=u\cos\alpha\ e^{-kt}$, we

get $$x=\frac{u\cos\alpha}{-k}e^{-kt}+C.$$

Initially when $t=0,\ x=0$; $\therefore\quad C=\frac{u\cos\alpha}{k}$.

$$\therefore\quad x=\frac{u\cos\alpha}{k}(1-e^{-kt}). \qquad \ldots(5)$$

Again integrating equation (4), *i.e.*

$$g+k\frac{dy}{dt}=(g+ku\sin\alpha)\,e^{-kt}.$$

we get $$gt+ky=-\frac{(g+ku\sin\alpha)}{k}e^{-kt}+D.$$

Initially when $t=0,\ y=0$; $\therefore\quad D=\frac{g+ku\sin\alpha}{k}$.

$$\therefore \qquad gt+ky=\frac{g+ku\sin\alpha}{k}(1-e^{-kt}). \qquad \ldots(6)$$

Equations (5) and (6) give the horizontal and vertical distances described by the particle in time t and these may be taken as the parametric equations of the trajectory. In order to get the cartesian equation of the trajectory, we have to eliminate the parameter t between (5) and (6).

From (5), $1-\dfrac{kx}{u\cos\alpha}=e^{-kt}$; $\therefore\ t=-\dfrac{1}{k}\log\left(1-\dfrac{kx}{u\cos\alpha}\right)$.

Putting for t and $1-e^{-kt}$ in (6), we get

$$y=\frac{-g}{k}t+\frac{g+ku\sin\alpha}{k^2}(1-e^{-kt}),$$

$$y=\frac{g}{k^2}\log\left(1-\frac{kx}{u\cos\alpha}\right)+\frac{g+ku\sin\alpha}{k^2}\cdot\frac{kx}{u\cos\alpha}$$

or $$y=\frac{g}{k^2}\log\left(1-\frac{kx}{u\cos\alpha}\right)+\frac{x}{ku\cos\alpha}\cdot(g+ku\sin\alpha). \qquad \ldots(7)$$

Exercise 7.

Ex. 1. *Prove that by a proper choice of axes the equation of the path can be put in the form* $y+ax=b\log x$.

(Punjab 55 ; Agra 59)

Putting $1-\dfrac{kx}{u\cos\alpha}=X$; $\quad\therefore\quad \dfrac{x}{u\cos\alpha}=\dfrac{1-X}{k}$.

Hence the equation (7) can be put as

$$y=\frac{g}{k^2}\log X+\frac{1-X}{k^2}(g+ku\sin\alpha),$$

$$y-\frac{g+ku\sin\alpha}{k^2}=\frac{g}{k^2}\log X-\frac{g+ku\sin\alpha}{k^2}X.$$

Putting $y-\dfrac{g+ku\sin\alpha}{k^2}=Y$, the above equation is

$$Y+\frac{g+ku\sin\alpha}{k^2}X=\frac{g}{k^2}\log Y$$

which is of the form $\qquad Y+aX=b\log X$.

Range and time of flight.

Ex. 2. *Show that if R be the range on the horizontal plane through the point of projection, then*

$$(uk \sin\alpha+g)\, kR+gu\cos\alpha \log\left(1-\frac{kR}{u\cos\alpha}\right)=0,$$

where u is the velocity of projection. Also show that if k is small, the range and the time of flight are approximately

$$\frac{2u^2\sin\alpha\cos\alpha}{g}-\frac{8}{3}k\frac{u^3\cos\alpha\sin^2\alpha}{g^2}.$$

and $\dfrac{2u\sin\alpha}{g}-\dfrac{2}{3}k\dfrac{u^2\sin^2\alpha}{g^2}$ *respectively.* (Agra 62)

Range. If the particle strikes the horizontal plane through the point of projection at a distance R, then the point $(R, 0)$ will satisfy the equation of trajectory *i e.* equation 7 of § 3.

$$y=\frac{g}{k^2}\log\left(1-\frac{kx}{u\cos\alpha}\right)+\frac{x}{ku\cos\alpha}(g+ku\sin\alpha).$$

$$\therefore\quad 0=\frac{g}{k^2}\log\left(1-\frac{kR}{u\cos\alpha}\right)+\frac{R}{ku\cos\alpha}(g+ku\sin\alpha).$$

Multiplying throughout by $k^2u\cos\alpha$, we get

$$gu\cos\alpha\log\left(1-\frac{kR}{u\cos\alpha}\right)+kR(g+ku\sin\alpha)=0. \quad ...(1)$$

Approximate value of R when k is small.

Expanding log in (1), we get

$$gu\cos\alpha\left[-\frac{kR}{u\cos\alpha}-\frac{k^2R^2}{2u^2\cos^2\alpha}-\frac{k^3R^3}{3u^3\cos^3\alpha}\cdots\right]$$
$$+kRg+k^2Ru\sin\alpha=0.$$

Cancel kR throughout.

$$\therefore\quad -g-\frac{kRg}{2u\cos\alpha}-\frac{1}{3}\frac{k^2R^2g}{u^2\cos^2\alpha}+g+ku\sin\alpha=0.$$

or $$\frac{Rg}{2u\cos\alpha}=u\sin\alpha-\frac{1}{3}\frac{kR^2g}{u^2\cos^2\alpha}.$$

$$\therefore\quad R=\frac{2u^2\sin\alpha\cos\alpha}{g}-\frac{2}{3}\frac{kR^2}{u\cos\alpha}. \qquad ...(2)$$

To a first approximation $R=\dfrac{2u^2\sin\alpha\cos\alpha}{g}$.

Putting the above value of R in R.H.S. of (2), we get the approximate value of R as

$$R=\frac{2u^2\sin\alpha\cos\alpha}{g}-\frac{2}{3}\frac{k}{u\cos\alpha}\cdot\frac{4u^4\sin^2\alpha\cos^2\alpha}{g^2}$$

or $$R=\frac{2u^2\sin\alpha\cos\alpha}{g}-\frac{8}{3}k\frac{u^3\cos\alpha\sin^2\alpha}{g^2}.$$

If U and V be the initial horizontal and vertical components of velocity of projection, then

$$U=u\cos\alpha \text{ and } V=u\sin\alpha$$

or $$R=\frac{2UV}{g}-\frac{8UV^2k}{3g^2}. \qquad \textbf{(Agra 62)}$$

Time of flight.

If t be the time of flight after which the particle strikes the horizontal plane through the point of projection, then during this time the vertical distance described by the particle is zero. From result (6) of § 3, we have

$$ky+gt=\frac{g+ku\sin\alpha}{k}(1-e^{-kt}).$$

Putting $y=0$, we get

$$gt=\frac{g+ku\sin\alpha}{k}\left[1-\left(1-kt+\frac{k^2t^2}{2!}-\frac{k^3t^3}{3!}\cdots\right)\right]$$

$$kgt=(g+ku\sin\alpha)\left[kt-\frac{k^2t^2}{2}+\frac{k^3t^3}{6}\cdots\right]$$

$$=kgt+k^2\left(u\sin\alpha.t-\frac{gt^2}{2}\right)+k^3\left(\frac{gt^3}{6}-u\sin\alpha\frac{t^2}{2}\right)\cdots$$

$$\left[u \sin \alpha - \frac{gt}{2} + k\left(\frac{gt^2}{6} - \frac{u \sin \alpha}{2} \cdot t\right)\right]$$

$$t = \frac{2u \sin \alpha}{g} + \frac{2k}{g}\left(\frac{gt^2}{6} - \frac{u \sin \alpha}{2} \cdot t\right) \ldots$$

To a first approximation, $t = \frac{2u \sin \alpha}{g}$.

Putting this value of t in the R.H.S. of (2), we get

$$t = \frac{2u \sin \alpha}{g} + \frac{2k}{g}\left(\frac{g}{6} \cdot \frac{4u^2 \sin^2 \alpha}{g^2} - \frac{u \sin \alpha}{2} \cdot \frac{2u \sin \alpha}{g}\right)$$

$$= \frac{2u \sin \alpha}{g} + \frac{2k}{g} \cdot \frac{u^2 \sin^2 \alpha}{g}\left(\frac{2}{3} - 1\right)$$

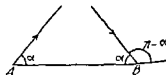
or $$t = \frac{2u \sin \alpha}{g} - \frac{2}{3} k \frac{u^2 \sin^2 \alpha}{g^2}.$$

Hence proved.

Ex. 3. *A particle of unit mass is projected with velocity u at an inclination α above the horizon in a medium whose resistance is k times the velocity. Show that its direction will again make an angle α with the horizon after a time*

$$\frac{1}{k} \log \left\{1 + \frac{2ku}{g} \sin \alpha\right\}.$$ **(Vikram 63 ; Cal. Hons. 62)**

If θ be the angle which the direction of motion makes with the horizon, then $\tan \theta = \frac{\dot{y}}{\dot{x}}$. ...(1)



Since the direction is to make an angle α with the horizon *i.e.* $\pi - \alpha$ with +ive direction of horizontal (measured anti-clockwise) *i.e.* $\theta = \pi - \alpha$. Putting $\theta = \pi - \alpha$ or $\tan \theta = -\tan \alpha$ in (1) and putting for $\dot{y}$ and $\dot{x}$ from (2) and (4) of § 1, we get

$$-\tan \alpha = \frac{(ku \sin \alpha + g)\, e^{-kt} - g}{ku \cos \alpha\, e^{-kt}}$$

or $$-ku \sin \alpha\, e^{-kt} = (ku \sin \alpha + g)\, e^{-kt} - g$$

or $$g e^{kt} = 2ku \sin \alpha + g$$

or
$$e^{kt}=1+\frac{2ku \sin \alpha}{g}.$$

$$\therefore \quad t=\frac{1}{k} \log \left(1+\frac{2ku \sin \alpha}{g}\right). \text{ Proved.}$$

Ex. 4. Greatest height and time for the same.

Suppose the particle is at its greatest height at time t; Then during this time its vertical component of velocity $\frac{dy}{dt}$ will be zero.

From (4) of § 3, $\frac{dy}{dt}=\frac{(ku \sin \alpha+g)\, e^{-kt}-g}{k}=0.$

$$\therefore \quad e^{kt}=\frac{ku \sin \alpha+g}{g}, \quad \therefore \quad t=\frac{1}{k} \log \left(1+\frac{ku \sin \alpha}{g}\right).$$

The greatest height is the value of y attained in time t given by the above

From result (6) of § 3, we have

$$gt+ky=\frac{g+ku \sin \alpha}{k}(1-e^{-kt}).$$

$$\therefore \quad \frac{g}{k} \log \left(1+\frac{ku \sin \alpha}{g}\right)+ky=\frac{g+ku \sin \alpha}{k}-\frac{g}{k}$$

or
$$ky=u \sin \alpha-\frac{g}{k} \log \left(1+\frac{ku \sin \alpha}{g}\right).$$

$$\therefore \quad y=\frac{u \sin \alpha}{k}-\frac{g}{k^2} \log \left(1+\frac{ku \sin \alpha}{g}\right). \text{ Proved.}$$

Deduction. Prove that the time to greatest height is less than half the time of flight.

Let t_1 be time of greatest height and t_2 be the time of flight ; then we have to prove that $t_1 < \frac{1}{2}t_2$ or $2t_1-t_2$ is $-$ive. Putting for t_1 and t_2 from Ex. 3 and 4, we have to prove that $\frac{2}{k} \log \left(1+\frac{ku \sin \alpha}{g}\right)-\left(\frac{2u \sin \alpha}{g}-\frac{2}{3}\frac{ku^2 \sin^2 \alpha}{g^2}\right)$

is $-$ive

or $$\frac{2}{k}\left(\frac{ku \sin \alpha}{g}-\frac{1}{2}\frac{k^2u^2 \sin^2 \alpha}{g^2}\right)-\left(\frac{2u \sin \alpha}{g}-\frac{2}{3}\frac{ku^2 \sin^2 \alpha}{g^2}\right)$$ is $-$ive

or $$-\frac{k^2u^2 \sin^2 \alpha}{g^2}-\frac{2}{3}\frac{ku^2 \sin^2 \alpha}{g^2} \text{ is } -\text{ive}.$$

Above is clearly true.

Ex. 5. *If the resistance vary as the velocity and the range on the horizontal plane through the point of projection is maximum, show that the angle σ which the direction of projection makes with the vertical is given by*

$$\frac{\lambda\,(1+\lambda \cos \alpha)}{\lambda+\cos \alpha}=\log\,(1+\lambda \sec \alpha),$$

where λ is the ratio of the velocity of projection to the terminal velocity. (Sagar 63; Agra 44 ; 63 ; Pb. 57)

If V be the terminal velocity, then $g-Vk=0$; $\therefore\ V=\frac{g}{k}$.

Also if u be the velocity of projection, then $\frac{u}{V}=\lambda$ given.

$$\therefore \quad u=V\lambda=\frac{g}{k}\lambda. \qquad ...(1)$$

Let us suppose that the direction of projection makes an angle θ with the horizontal. Hence by the condition $\theta=90-\sigma$.

Taking θ in place of α in the results of § 3, we know that $x=\frac{u \cos \theta}{k}\ (1-e^{-kt})$ and $gt+ky=\frac{g+ku \sin \theta}{k}\ (1-e^{-kt})$.

[Results (5) and (6) of § 3.]

Now range is the value of x obtained in time t during which $y=0$.

$$\therefore \quad R=\frac{u \cos \theta}{k}(1-e^{-kt}), \qquad ...(2)$$

where $$gt=\frac{g+ku\sin\theta}{k}(1-e^{-kt}). \qquad ...(3)$$

Range is maximum, when $\frac{dR}{d\theta}=0$.

Differentiating (2) and (3) w. r. t. θ, we get

$$0=\frac{-u\sin\theta}{k}(1-e^{-kt})+\frac{u\cos\theta}{k}ke^{-kt}.\frac{dt}{d\theta}, \qquad ...(4)$$

where $$g.\frac{dt}{d\theta}=u\cos\theta(1-e^{-kt})+\frac{g+ku\sin\theta}{k}ke^{-kt}.\frac{dt}{d\theta}. \qquad ...(5)$$

$\therefore$ $[g-(g+ku\sin\theta)e^{-kt}]\frac{dt}{d\theta}=u\cos\theta(1-e^{-kt})$ from (5)

and $[u\cos\theta\, e^{-kt}]\frac{dt}{d\theta}=\frac{u\sin\theta}{k}(1-e^{-kt})$ from (4).

Dividing the above in order to eliminate $\frac{dt}{d\theta}$, we get

$$\frac{g-(g+ku\sin\theta)e^{-kt}}{u\cos\theta.e^{-kt}}=\frac{k\cos\theta}{\sin\theta}$$

or $$g\sin\theta=e^{-kt}(uk\cos^2\theta+uk\sin^2\theta+g\sin\theta)$$

or $$\frac{g\sin\theta}{uk+g\sin\theta}=e^{-kt}$$

or $$t=-\frac{1}{k}\log\frac{g\sin\theta}{uk+g\sin\theta}.$$

Putting for t and e^{-kt} from above in (3), we get

$$-\frac{g}{k}\log\frac{g\sin\theta}{uk+g\sin\theta}=\frac{g+ku\sin\theta}{k}\left(1-\frac{g\sin\theta}{uk+g\sin\theta}\right).$$

Now put $\theta=90-\alpha$ and $u=\frac{g}{k}\lambda$, *i e.* $uk=g\lambda$.

$$\therefore\ \frac{g}{k}\log\frac{g\lambda+g\cos\alpha}{g\cos\alpha}=\frac{g+g\lambda\cos\alpha}{k}\left(1-\frac{g\cos\alpha}{g\lambda+g\cos\alpha}\right)$$

or $$\log(1+\lambda\sec\alpha)=(1+\lambda\cos\alpha)\left[\frac{\lambda}{\cos\alpha+\lambda}\right].$$ **Proved.**

Ex. 6. *A particle acted on by gravity is projected in a medium; the resistance of which varies as the velocity; show that its acceleration retains a fixed direction and diminishes without limit to zero.* (Vikram 64)

If θ be the direction of acceleration with the horizontal,

then $$\tan\theta=\frac{\ddot{y}}{\ddot{x}}=\frac{-g-k\frac{dy}{dt}}{-k\frac{dx}{dt}} \text{ from (1) and (3) of § 3}$$

$$=\frac{(g+ku\sin\alpha)\,e^{-kt}}{ku\cos\alpha\,e^{-kt}} \text{ from (4) and (2) of § 3}$$

or $$\tan\theta=\frac{g+ku\sin\alpha}{ku\cos\alpha}.$$

Since the value of $\tan\theta$ is independent of t, hence it retains a fixed direction. Again if f be the magnitude of acceleration, then

$$f=\sqrt{(\ddot{x}^2+\ddot{y}^2)}=\sqrt{[(g+ku\sin\alpha)^2+k^2u^2\cos^2\alpha]}\,.\,e^{-kt}$$

or $$f=\frac{g^2+k^2u^2+2gku\sin\alpha}{e^{kt}}.$$

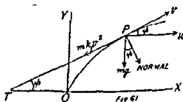
As t increases e^{kt} continually increases and consequently f decreases as t tends to ∞ thus f tends to zero. **Proved.**

§ 4. Trajectory in a resisting medium when resistance varies as square of velocity

A particle is projected under gravity and a resistance equal to mk (velocity)²; to discuss the motion. (Agra 64, 54)

Let P be the position of the particle where its velocity is v and horizontal component of this velocity be denoted by u, so that

$$u=v\cos\psi. \quad ...(1)$$



Here we shall write down the equations of motion parallel to x-axis and along the normal. (Note carefully)

Since u is the velocity parallel to x-axis, so the acceleration parallel to x-axis is $\frac{du}{dt}$. The forces acting on the particle are its weight mg acting vertically downwards and resistance mkv^2 along the tangent as shown. Hence using $P=mf$, the equations of motion parallel to x-axis and along the normal are

$$m.\frac{du}{dt}=-mkv^2 \cos\psi \quad \text{or} \quad \frac{du}{dt}=-kv^2 \cos\psi \qquad ...(2)$$

and $$m\frac{v^2}{\rho}=mg\cos\psi \quad \text{or} \quad \frac{v^2}{\rho}=g\cos\psi. \qquad ...(3)$$

We shall aim at finding the intrinsic *i.e.* s and ψ relation of the path by the help of equations (2) and (3).

Now $\rho=-\frac{ds}{d\psi}$ (as ψ is decreasing).

Now from (2), we have $\frac{du}{dt}=-kv.v\cos\psi=-k.\frac{ds}{dt}.u$ by (1)

$$\therefore \quad \frac{du}{u}=-k\,ds.$$

Integrating, we get $\log u=-ks+A$.

Initially when $s=0$, suppose $u=u_0$.

$\therefore \quad A=\log u_0. \quad \therefore \quad \log\frac{u}{u_0}=-ks$ or $u=u_0e^{-ks}$ ...(4)

Again from (3), we have $g\cos\psi=v^2\left(-\frac{d\psi}{ds}\right)$

or $$g\cos\psi=v.\frac{ds}{dt}\left(-\frac{d\psi}{ds}\right)=-v\frac{d\psi}{dt}$$

or $$g\cos\psi=-v.\frac{d\psi}{du}.\frac{du}{dt}$$

or $$g \cos \psi = -v \frac{d\psi}{du} . (-kv^2 \cos \psi) \text{ by (2)}$$

or $$g = kv^3 \frac{d\psi}{du} = k \,.\, u^3 \sec^3 \psi \frac{d\psi}{du} \quad \because \; u = v \cos \psi.$$

$$\therefore \quad \frac{du}{u^3} = \frac{k}{g} \sec^3 \psi \, d\psi.$$

Integrating, we get

$$-\frac{1}{2u^2} = \frac{k}{g}\left[\frac{\sec \psi \tan \psi}{2} + \frac{1}{2} \log (\sec \psi + \tan \psi)\right] + A$$

or $$\frac{1}{u_0^2} e^{2ks} = -\frac{k}{g} [\sec \psi \tan \psi + \log (\sec \psi + \tan \psi)] + \text{constant}$$

by (4), ...(5)

Again $\frac{v^2}{\rho} = g \cos \psi$; $\therefore \quad \rho = \frac{v^2}{g \cos \psi} = \frac{u^2}{g \cos^3 \psi}$

$\because \; v \cos \psi = u.$

$$\therefore \; \rho = \frac{u_0 e^{-2ks}}{g} \sec^3 \psi. \qquad \text{...(6)}$$

Above is the intrinsic equation of the path. In order to find the constant of integration we shall use the initial conditions.

Note. $\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx.$

Note. Since $u = u_0 e^{-ks} = \frac{u_0}{e^{ks}}$ by (4), we observe that as s increases, e^{ks} also increases and u decreases continually. Hence as s tends to infinity u tends to zero showing thereby that the particle tends to move vertically.

§ 5. Trajectory in a resisting medium when resistance varies as nth power of velocity.

Proceeding exactly as in § 4, we have

u = horizontal component of velocity $= v \cos \psi$ ...(1)

The equations of motion parallel to x-axis and along the normal are

$$\frac{du}{dt}=-kv^n \cos\psi \quad ...(1) \text{ and } \frac{v^2}{\rho}=g\cos\psi ; \quad ...(3)$$

where $\rho=-\dfrac{ds}{d\psi}$ (ψ decreasing as s increases).

From (3), $v^2\left(-\dfrac{d\psi}{ds}\right)=g\cos\psi$

or $v\cdot\dfrac{ds}{dt}\left(-\dfrac{d\psi}{ds}\right)=g\cos\psi$

or $-v\cdot\dfrac{d\psi}{dt}=g\cos\psi$ or $-v\cdot\dfrac{d\psi}{du}\cdot\dfrac{du}{dt}=g\cos\psi$

$$-v\frac{d\psi}{du}(-kv^n\cos\psi)=g\cos\psi \text{ by (1)}$$

or $kv^{n+1}\dfrac{d\psi}{du}=g$ or $k\cdot\dfrac{u^{n+1}}{\cos^{n+1}\psi}\cdot\dfrac{d\psi}{du}=g$, $\because v\cos\psi=u$.

$$\therefore \quad \frac{du}{u^{n+1}}=\frac{k}{g}\sec^{n+1}\psi\, d\psi.$$

Integrating both sides, we get

$$-\frac{1}{nu^n}=\frac{k}{g}\int\sec^{n+1}\psi\, d\psi+A$$

or $\dfrac{1}{u^n}=-n\dfrac{k}{g}\displaystyle\int\sec^{n+1}\psi\, d\psi+\text{constant}$

or $\dfrac{1}{(v\cos\psi)^n}=-\dfrac{nk}{g}\displaystyle\int\sec^{n+1}\psi\, d\psi+\text{constant}.$ ...(4)

Above gives us the velocity v at any point ψ.

Note. It is to be observed that in § 4 and § 5 we have written equations of motion parallel to x-axis and along the normal though usually we write the equations of motion along the tangent and normal.

Exercise 8.

Ex. 1. (a) *If ρ and ρ' be the radii of curvature at two points at equal arcural distances from the vertex, ψ, ψ' the inclinations to the horizon of tangents, prove that $\rho\rho' \cos^3 \psi \cos^3 \psi' = \rho_0^2$ where ρ_0 is the radius of curvature at the vertex. Resistance varying as square of the velocity.*

Let u be the horizontal component of velocity at P,

i.e. $u = v \cos \psi$.

Proceeding as in § 4, and resolving along the parallel to x-axis and normally, we get

$$\frac{du}{dt} = -kv^2 \cos \psi. \quad \text{(Refer fig. § 5.)}$$

$$\frac{v^2}{\rho} = g \cos \psi; \quad \therefore \quad \rho = \frac{v^2}{g \cos \psi} = \frac{u^2}{g \cos^3 \psi}. \quad \ldots(1)$$

Now $\dfrac{du}{dt} = -kv \cdot v \cos \psi = -k \dfrac{ds}{dt} u.$

or $\dfrac{du}{u} = -k\, ds; \quad \therefore \quad \log u = -ks + A.$

Initially when $s = 0$, let $u = u_0$; $\therefore$ $A = \log u_0$.

$$\therefore \log \frac{u}{u_0} = -ks \quad \text{or} \quad u = u_0 e^{-ks}. \quad \ldots(2)$$

Putting for u from (2) in (1), we get

$\rho \cos$ [illegible] e^{-2ks}. ...(3)

Putting $s = -s$ ar[illegible] $\psi = \psi'$, [illegible]

where v is the velocity and k a constant. If the initial velocity is V and the square of $k\frac{V^2}{g}$ can be neglected, show that the particle reaches its highest point in time $\frac{V}{g}-\frac{kV^3}{3g^2}$ and that the greatest altitude reached is $\frac{V^2}{2g}-\frac{kV^4}{4g^2}$.

If the initial velocity, in addition to the vertical component V, has a small horizontal component u and the resistance follows the same law, show that when particle returns to the original level, its horizontal velocity is approximately.

$$ue^{-kV^2/g}.$$

The equation of motion is

$$m\frac{d^2x}{dt^2}=-mg-mkv^2 \quad \text{or} \quad v\frac{dv}{dx}=-g-kv^2=-k\left(v^2+\frac{g}{k}\right).$$

$$\therefore \frac{2v\,dv}{v^2+\frac{g}{k}}=-2k\,dx.$$

Integrating, $\log\left(v^2+\frac{g}{k}\right)=-2kx+A.$

Initial condition gives $x=0,\ v=V$; $\therefore\ A=\log.\left(V^2+\frac{g}{k}\right).$

$$\therefore \log\left(v^2+\frac{g}{k}\right)=-2kx+\log\left(V^2+\frac{g}{k}\right).$$

Now if x be the greatest height, then $v=0$ at that point

$$\therefore \log\frac{g}{k}=-2kx+\log\left(V^2+\frac{g}{k}\right)$$

or
$$\log\left(V^2+\frac{g}{k}\right)-\log\frac{g}{k}=2kx$$

or
$$2kx=\log\left(1+\frac{V^2}{g/k}\right)=\log\left(1+\frac{kV^2}{g}\right)$$

or $$2kx=\frac{kV^2}{g}-\frac{1}{2}\left(\frac{kV^2}{g}\right)^2; \quad \therefore \quad x=\frac{V^2}{2g}-\frac{kV^4}{4g^2}. \qquad ...(1)$$

In order to find the time we write the equation of motion as
$$\frac{dv}{dt}=-g-kv^2 \quad \text{or} \quad \frac{dv}{g+kv^2}=-dt.$$

Integrating, $$\frac{1}{\sqrt{k}}\cdot\frac{1}{\sqrt{g}}\tan^{-1}\left\{\sqrt{\left(\frac{k}{g}\right)}\, v\right\}=-t+C.$$

Initially when $t=0$, $v=V$.
$$\therefore \quad \frac{1}{\sqrt{(kg)}}\tan^{-1}\sqrt{\left(\frac{k}{g}\right)}\, V=C.$$

Putting for C in the above, we get
$$t=\frac{1}{\sqrt{(kg)}}\left\{\tan^{-1}\sqrt{\left(\frac{k}{g}\right)}\, V-\tan^{-1}\sqrt{\left(\frac{k}{g}\right)}\, v\right\}.$$

For the highest point $v=0$;
$$\therefore \quad t=\frac{1}{\sqrt{(kg)}}\tan^{-1}\left\{\sqrt{\left(\frac{k}{g}\right)}\, V\right\}$$

or $$t=\frac{1}{\sqrt{(kg)}}\left\{\sqrt{\left(\frac{k}{g}\right)}\, V-\frac{1}{3}\left(\frac{k}{g}\right)^{3/2}V^3+...\right\}.$$

Note that we have to reject square of $\frac{kV^2}{g}$, etc.
$$\therefore \quad t=\frac{V}{g}-\frac{kV^3}{3g^2}.$$

2nd Case. Here the particle has initially horizontal velocity also and hence the particle will not move vertically upwards but will describe a trajectory. If v be the velocity at any instant along the tangent inclined at an angle ψ to the horizontal, then resistance obeys the same law of kv^2.
$\therefore$ resistance along horizontal is $kv^2 \cos\psi$. Hence the equation of motion along the horizontal is
$$\frac{d^2x}{dt^2}=-kv^2\cos\psi \quad (x \text{ along the horizontal})$$

or $$\frac{d^2x}{dt^2}=-k\frac{ds}{dt}\cdot\frac{ds}{dt}\cdot\frac{dx}{ds}, \qquad \therefore \cos\psi=\frac{dx}{ds}.$$

or $$\frac{d^2x}{dt^2}=-k\frac{ds}{dt}\cdot\frac{dx}{dt} \text{ or } \frac{\ddot{x}}{\dot{x}}=-k\dot{s}.$$

Integrating, we get $\log \dot{x}=-ks+A$.

Initially when $s=0$, $\dot{x}$=horizontal velocity$=u$;

$$\therefore \quad A=\log u.$$

$$\therefore \quad \log \dot{x}=-ks+\log u \quad \text{or} \quad \log\frac{\dot{x}}{u}=-ks.$$

or $$\dot{x}=ue^{-ks}. \qquad \ldots(2)$$

Now we have to find the value of horizontal velocity $\dot{x}$ when the particle reaches the horizontal plane again and hence we should know the value of s when the particle returns. One thing is to be noted here that u the initial horizontal velocity of the particle is very small and it will go on decreasing on account of resistance and hence as an approximation, we may take s the total arc of the trajectory described before reaching the plane to be equal to twice the greatest height reached.

$$\therefore \quad s=2\left(\frac{V^2}{2g}-\frac{kV^4}{4g^2}\right) \text{ by (1)}$$

or $$s=\frac{V^2}{g}-\frac{kV^4}{2g^2} \quad \text{or} \quad ks=\frac{kV^2}{g}-\frac{k^2V^4}{2g^2}$$

or $$ks=\frac{kV^2}{g} \text{ as squares of } \frac{kV^2}{g} \text{ are to be neglected.}$$

Putting the value of ks in (2), we get $\dot{x}=ue^{-\frac{kV^2}{g}}$. Proved.

Ex. 2. *If the resistance per unit mass is* $g\left(\frac{v}{V}\right)^2$, *prove that* $\frac{du}{ds}=-\frac{g}{V^2}u$, $\frac{d\psi}{ds}=\frac{g}{u^2}\cos^3\psi$, *where* u *is the horizontal component of velocity.*

If v be the velocity at any point, then $u = v\cos\psi$. ...(1)

Also equations of motion along the horizontal and normal are $\frac{du}{dt} = -g\frac{v^2}{V^2}\cos\psi$ ∵ u is the horizontal velocity

and $\frac{v^2}{\rho} = g\cos\psi$.

$$\frac{du}{dt} = -\frac{g}{V^2}v.(v\cos\psi) = -\frac{g}{V^2}.\frac{ds}{dt}u \text{ by (1)}$$

$$\therefore \frac{du}{ds} = -\frac{g}{V^2}u.$$ Proved.

Also $\rho = \frac{v^2}{g\cos\psi} = \frac{u^2}{g\cos^3\psi}$; $\therefore \frac{ds}{d\psi} = \frac{u^2}{g\cos^3\psi}$.

$$\therefore \frac{d\psi}{ds} = \frac{g}{u^2}\cos^3\psi.$$ Proved.

Ex. 3. *Prove that if the resistance vary as square of velocity, then*

$$V^2\left[\frac{1}{u^2} - \frac{1}{u_0^2}\right] = \sec\psi_0\tan\psi_0 - \sec\psi\tan\psi + \log\frac{\sec\psi_0 + \tan\psi_0}{\sec\psi + \tan\psi}$$

where u_0 and ψ_0 are the initial values of u and ψ, V the terminal velocity.

It is exactly a reproduction of § 4.

Let the resistance be kv^2 so that acceleration is $g - kv^2$ which will be zero if V be the velocity, so that $g - kV^2 = 0$.

$\therefore V^2 = \frac{g}{k}$. This V is called terminal velocity.

Let v be the velocity at any position and ψ be the inclination of the tangent to the horizontal. If u be the horizontal component of this velocity, then $v\cos\psi = u$. ...(1)

Resolving along the horizontal and normal as in § 4,

we have $$\frac{du}{dt} = -kv^2\cos\psi.$$ ...(2)

and $$\frac{v^2}{\rho}=g\cos\psi. \qquad ...(3)$$

Also $\rho=-\dfrac{ds}{d\psi}$ (as ψ decreases when s increases).

$$\therefore\quad g\cos\psi=\frac{v^2}{\rho}=v.\left(\frac{ds}{dt}\right)\left(-\frac{\psi d}{ds}\right)$$

or $$g\cos\psi=-v\frac{d\psi}{dt}=-v\left(\frac{d\psi}{du}\right)\left(\frac{du}{dt}\right)$$

or $$g\cos\psi=-v\frac{d\psi}{du}(-kv^2\cos\psi)\ \text{ by (2).}$$

$$\therefore\quad g=kv^3\frac{d\psi}{du}=k.u^3\sec^3\psi\frac{d\psi}{du}\qquad\because\ v\cos\psi=u.$$

$$\therefore\quad \frac{g}{k}.\frac{du}{u^3}=\sec^3\psi\,d\psi.\qquad \text{Put } \frac{g}{k}=V^2$$

$$\therefore\quad V^2\int_{u_0}^{u}\frac{du}{u^3}=\int_{\psi_0}^{\psi}\sec^3\psi\,d\psi$$

or $$\frac{V^2}{-2}\left[\frac{1}{u^2}\right]_{u_0}^{u}=\left[\frac{\sec\psi\tan\psi}{2}+\frac{1}{2}\log(\sec\psi+\tan\psi)\right]_{\psi_0}^{\psi}$$

or $$-V^2\left[\frac{1}{u^2}-\frac{1}{u_0^2}\right]=[(\sec\psi\tan\psi-\sec\psi_0\tan\psi_0)]$$
$$+\log(\sec\psi+\tan\psi)-\log(\sec\psi_0+\tan\psi_0)].$$

Multiplying throughout by $-$ive sign, we get

$$V^2\left[\frac{1}{u^2}-\frac{1}{u_0^2}\right]=(\sec\psi_0\tan\psi_0-\sec\psi\tan\psi)+\log\frac{\sec\psi_0+\tan\psi_0}{\sec\psi+\tan\psi}.$$

Proved.

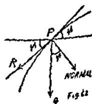
Ex. 4. *Prove that in the motion of a projectile in a resisting medium the equation* $\dfrac{d^2y}{dx^2}=-\dfrac{g}{u^2}$ *is satisfied whatever the law of resistance, u being the horizontal component of velocity, the axes of x and y being horizontal and vertically upwards.*

If the resistance is constant and equal to kg, show that the velocity at any point is given by

$$v\,(1-\sin\psi)^k = u_0\,(\cos\psi)^{k-1}$$

ψ being the slope and u_0 the velocity at the highest point.

(Punjab 54)



1st Part. Let R be the resistance per unit mass; then resolving horizontally and vertically, we have

$$\frac{d^2x}{dt^2} = -R\cos\psi,\quad \frac{d^2y}{dt^2} = -g - R\sin\psi,$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}.$$

Differentiating both sides w. r. t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\dot{y}}{\dot{x}}\right) = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)\cdot\frac{dt}{dx}$$

or $$\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{x}^2}\cdot\frac{1}{\dot{x}}$$

or, $$\frac{d^2y}{dx^2} = \frac{\dot{x}\,(-g-R\sin\psi)-\dot{y}\,(-R\cos\psi)}{\dot{x}^3}$$

$$= \frac{-g\dot{x}-R\,(\dot{x}\sin\psi-\dot{y}\cos\psi)}{\dot{x}^3}.$$

But $\tan\psi = \frac{\dot{y}}{\dot{x}}$; $\therefore\ \dot{x}\sin\psi - \dot{y}\cos\psi = 0$

$$\therefore\ \frac{d^2y}{dx^2} = -\frac{g\dot{x}}{\dot{x}^3} = -\frac{g}{\dot{x}^2}.$$

But $\dot{x}$ is the horizontal component of velocity is u given.

$$\therefore\quad \frac{d^2y}{dx^2} = -\frac{g}{u^2}.$$ **Proved.**

2nd Part. Here the resistance is constant $=kg$ and we are to find a relation between v and ψ. Let u stand for horizontal component of velocity, then $v\cos\psi = u$. Also at

the highest point, where $\psi=0$, the velocity is u_0 which will be horizontal, so that when $\psi=0$, $u=u_0$.

Equations of motion along the horizontal and normal as usual are

$$\frac{du}{dt}=-kg\cos\psi, \quad ...(1) \quad \frac{v^2}{\rho}=g\cos\psi. \quad ...(2)$$

Also $\rho=-\dfrac{ds}{d\psi}$ (as ψ decreases with the increase of s).

$$v^2\left(-\frac{d\psi}{ds}\right)=g\cos\psi \quad \text{or} \quad v.\frac{ds}{dt}\left(-\frac{d\psi}{ds}\right)=g\cos\psi$$

or $$v\frac{d\psi}{dt}=-g\cos\psi \quad \text{or} \quad v\frac{d\psi}{du}.\frac{du}{dt}=-g\cos\psi$$

or $\dfrac{u}{\cos\psi}\dfrac{d\psi}{du}(-kg\cos\psi)=-g\cos\psi$ by (1) and $\because v\cos\psi=u$,

$$\therefore \quad -ku\frac{d\psi}{du}=-\cos\psi \quad \text{or} \quad \frac{du}{u}=k\sec\psi\, d\psi.$$

$$\therefore \quad \log u=k\log(\sec\psi+\tan\psi)+A.$$

At the highest point where $\psi=0$, $u=u_0$ given.

$$\therefore \quad A=\log u_0.$$

$$\therefore \quad \log\frac{u}{u_0}=\log(\sec\psi+\tan\psi)^k$$

or $$u=u_0(\sec\psi+\tan\psi)^k$$

or $$v\cos\psi=u_0\frac{(1+\sin\psi)^k}{(\cos\psi)^k}.$$

$$\therefore \quad v(1-\sin\psi)^k=u_0\frac{(1-\sin^2\psi)^k}{(\cos\psi)^{k+1}}=u_0\frac{\cos^{2k}\psi}{(\cos\psi)^{k+1}}$$

or $$v(1-\sin\psi)^k=u_0(\cos\psi)^{k-1}.$$ Proved.

Ex. 5. *If a heavy particle describes a path given by* $\cos\psi=f(\rho\cos\psi)$, *show that the law of resistance is*

$$Rv\frac{df}{dv}=-g\sqrt{(1-f^2)}\frac{d}{dv}(vf), \text{ where } f=f\left(\frac{v^2}{g}\right).$$

$$\therefore \quad \frac{du}{u^4}=\frac{k}{g}(1+\tan^2\psi)\sec^2\psi\, d\psi$$

Integrating, $-\dfrac{1}{3u^3}=\dfrac{k}{g}\left(\tan\psi+\dfrac{\tan^3\psi}{3}\right)+A$.

When $\psi=0$, *i.e.* the particle is moving horizontally, let the horizontal velocity be u_0 ; $\therefore \quad A=-\dfrac{1}{3u_0^3}$.

$$\therefore \quad \frac{1}{3}\left(\frac{1}{u_0^3}-\frac{1}{u^3}\right)=\frac{k}{g}\left(\tan\psi+\frac{\tan^3\psi}{3}\right).$$

Now $f=kv^3=k\dfrac{u^3}{\cos^3\psi}$; $\therefore \quad \dfrac{1}{u^3}=\dfrac{k}{f\cos^3\psi}$.

Also $\qquad \dfrac{1}{u_0^3}=\dfrac{k}{f_0}$ at $\psi=0$.

$$\therefore \quad \frac{k}{f_0}-\frac{k}{f\cos^3\psi}=3\frac{k}{g}\left(\tan\psi+\frac{\tan^3\psi}{3}\right)$$

or
$$\frac{1}{f\cos^3\psi}=\frac{1}{f_0}-\frac{3}{g}\frac{(3\sin\psi\cos^2\psi+\sin^3\psi)}{3\cos^3\psi}.$$

Also we are given that when $\psi=\alpha$, $f=f$, when $\psi=-\alpha$, $f=f'$.

$$\therefore \quad \frac{1}{f\cos^3\alpha}=\frac{1}{f_0}-\frac{1}{g}\frac{\sin\alpha\,(3-2\sin^2\alpha)}{\cos^3\alpha} \qquad ...(5)$$

and
$$\frac{1}{f'\cos^3\alpha}=\frac{1}{f_0}+\frac{1}{g}\frac{\sin\alpha\,(3-2\sin^2\alpha)}{\cos^3\alpha}, \qquad ...(6)$$

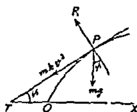
putting $-\alpha$ for α.

Adding, we get $\dfrac{1}{f}+\dfrac{1}{f'}=\dfrac{2}{f_0}\cos^3\alpha$ and subtracting, we get

$$\frac{1}{f'}-\frac{1}{f}=\frac{2\sin\alpha}{g}(3-2\sin^2\alpha). \qquad \text{Proved}$$

§ 6. Motion on a smooth curve under resistance.

A bead moves on a smooth wire in a vertical plane under the resistance which varies as square of the velocity ; to discuss the motion. (Indore 66; Agra 49, 59)



Here in addition to the resistance mkv^2 along the tangent and the weight mg vertically downwards, we will have the reaction R along the normal. Here it will be convenient to write down the equations of motion along the tangent and normal.

$$mv\frac{dv}{ds}=-mg\sin\psi-mkv^2. \quad ...(1)$$

and
$$m\frac{v^2}{\rho}=mg\cos\psi-R. \quad ...(2)$$

From (1), we have

$$\frac{1}{2}\frac{dv^2}{ds}+kv^2=-g\sin\psi.$$

or
$$\frac{dv^2}{ds}+2kv^2=-2g\sin\psi.$$

or
$$\frac{dv^2}{d\psi}\cdot\frac{d\psi}{ds}+2kv^2=-2g\sin\psi.$$

But
$$\rho=\frac{ds}{d\psi}. \quad ...(3)$$

$$\therefore\ \frac{dv^2}{d\psi}+2k\rho v^2=-2g\rho\sin\psi. \quad ...(4)$$

If the equation of the curve be given, then ρ can be known in terms of ψ and the above equation is linear and can be integrated to give us the value of v^2 in terms of ψ.

If the curve be a circle of radius a, then $\rho=a$ and the equation (3) becomes (Agra 50, 55, 57, 61)

$$\frac{dv^2}{d\psi}+2kav^2=-2ag\sin\psi.$$

It is linear and I.F. $=e^{2ka\psi}$.

Multiplying both sides by I. F. and integrating, we get

$$v^2 . e^{2ak\psi} = -2ag \int e^{2ak\psi} . \sin\psi \, d\psi + A.$$

$$\therefore \quad v^2 . e^{2ak\psi} = -\frac{2ag}{1+4a^2k^2} [e^{2ak\psi} (2ak \sin\psi - \cos\psi)] + A.$$

or $$v^2 = -\frac{2ag}{1+4a^2k^2} [2ak \sin\psi - \cos\psi] + Ae^{-2ak\psi}.$$

Above equation gives us v^2 in terms of ψ.

Another form. From result (3),

$$\frac{dv^2}{ds} + 2kv^2 = -2g \sin\psi.$$

Above is linear and I. F.$=e^{2ks}$. Multiplying throughout by e^{2ks} and integrating, we get

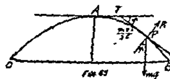
$$v^2 . e^{2ks} = -2g \int e^{2ks} \sin\psi \, ds.$$

Exercise 9.

Ex. 1. *A heavy bead of mass m slides on a smooth wire in the shape of a cycloid whose axis is vertical and vertex upwards, in a medium whose resistance is* $m\dfrac{v^2}{2c}$ *and the distance of the starting point from the vertex is c. Show that the time of descent to the cusp is* $\sqrt{\left\{\dfrac{8a(4a-c)}{gc}\right\}}$.

(Agra 46, 49, 54, 57, 64)

A is the vertex and B the cusp where length of $AB=4a$, s being measured from A. The intrinsic equation of cycloid is $s=4a \sin\psi$.



The forces acting on the particle are resistance $\dfrac{mv^2}{2c}$, weight mg and normal reaction R.

The tangential equation of motion is

$$mv\frac{dv}{ds} = mg \sin\psi - m\frac{v^2}{2c}$$

or $$\frac{1}{2}\frac{dv^2}{ds}=g\cdot\frac{s}{4a}-\frac{v^2}{2c}$$

or $$\frac{dv^2}{ds}+\frac{v^2}{c}=\frac{gs}{2a}.$$

Above is a linear differential equation and

$$\text{I. F.}=e^{\int \frac{1}{c}\,ds}=e^{s/c}.$$

Multiplying both sides by I. F. and integrating, we get

$$v^2\cdot e^{s/c}=\int \frac{g}{2a}\,s\cdot e^{s/c}\,ds+A$$

or $$v^2\cdot e^{s/c}=\frac{g}{2a}\left[s\cdot(ce^{s/c})-\int ce^{s/c}\cdot 1\,ds\right]+A$$

or $$v^2\cdot e^{s/c}=\frac{g}{2a}\left[cs\cdot e^{s/c}-c^2\cdot e^{s/c}\right]+A.$$

Initially when $s=c$ (given), $v=0$; $\therefore\ A=0$.

$$\therefore\quad v^2\cdot e^{s/c}=\frac{g}{2a}\cdot cece^{s/c}\,(s-c).$$

$$\therefore\quad v=\frac{ds}{dt}=\sqrt{\left(\frac{gc}{2a}\right)}\sqrt{(s-c)}$$

$$\therefore\quad \int_c^{4a}\frac{ds}{\sqrt{(s-c)}}=\int_{t=0}^{T}\sqrt{\left(\frac{gc}{2a}\right)}\,dt,$$

where T stands for the time when a particle comes to the cusp at B where $s=4a$.

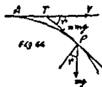
$$\therefore\quad 2\left[\sqrt{(s-c)}\right]_c^{4a}=\sqrt{\left(\frac{gc}{2a}\right)}\cdot T;\quad \therefore\quad T=2\sqrt{\left(\frac{2a}{gc}\right)}\sqrt{(4a-c)}$$

or $$T=\sqrt{\left\{\frac{8a\,(4a-c)}{gc}\right\}}.$$

Ex. 2. *If the resistance of the air to a particle's motion be n times its weight and the particle be projected horizontally with velocity V, show that the velocity of the particle when it is moving at an inclination ϕ to the horizontal is*

$$V\,(1-\sin\phi)^{(n-1)/2}\,(1+\sin\phi)^{-(n+1)/2}.$$

(Agra 58; Delhi Hon's. 59)



Since resistance $= n$. weight of particle $= mng$. Equations of motion along the tangent and normal are

$$mv\frac{dv}{ds}=mg\sin\psi-mng.$$

and
$$m\frac{v^2}{\rho}=mg\cos\psi$$

or
$$v\frac{dv}{ds}=g(\sin\psi-n), \quad ...(1)$$

and
$$\frac{v^2}{\rho}=g\cos\psi. \quad ..(2)$$

From the answer it is clear that we should establish a relation between v and ψ.

From (1), $v\,dv=g(\sin\psi-n)\dfrac{ds}{d\psi}.d\psi.$

But $\dfrac{ds}{d\psi}=\rho=\dfrac{v^2}{g\cos\psi}$ by (2).

$$\therefore\quad v\,dv=g(\sin\psi-n)\frac{v^2}{g\cos\psi}.d\psi$$

or
$$\frac{dv}{v}=(\tan\psi-n\sec\psi)\,d\psi.$$

Integrating, $\log v=\log\sec\psi-n\log(\sec\psi+\tan\psi)+A.$

Initially when $\psi=0$, $v=V$. $\therefore$ $A=\log V$.

$$\therefore\quad \log\frac{v}{V}=\log\sec\psi-\log\left(\frac{1+\sin\psi}{\cos\psi}\right)^n$$

or
$$\frac{v}{V}=\sec\psi.\frac{\cos^n\psi}{(1+\sin\psi)^n}$$

or
$$v=V\frac{\cos^{n-1}\psi}{(1+\sin\psi)^n}=V\frac{(1-\sin^2\psi)^{(n-1)/2}}{(1+\sin\psi)^n}$$

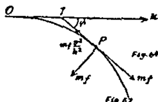
or
$$v=V(1-\sin\psi)^{(n-1)/2}(1+\sin\psi)^{\frac{n-1}{2}-n}.$$

Hence the velocity when $\psi=\phi$ is given by

$$v=V(1-\sin\phi)^{(n-1)/2}(1+\sin\phi)^{-(n+1)/2}.$$

Ex. 3. *If a particle of mass m be acted upon by equal constant forces mf tangentially and normally to the path and if the resistance be* $mf\dfrac{v^2}{k^2}$, *prove that intrinsic equation of the path is* $k^2(e^{2fs/k^2}-1)=u^2(e^{2\psi}-1)$, *where u is the velocity of projection.*

Here the particle is not projected in a vertical plane. Hence we shall not take into consideration the weight of the particle in writing down the equations of motion



The equations of motion of the particle along the tangent and normal are

$$mv\frac{dv}{ds}=mf-mf\cdot\frac{v^2}{k^2} \text{ and } m\frac{v^2}{\rho}=mf$$

or $$\frac{1}{2}\frac{dv^2}{ds}+f\frac{v^2}{k^2}=f \quad ...(1) \text{ and } \frac{v^2}{f}=\rho=\frac{ds}{d\psi}. \qquad ...(2)$$

As ψ increases when s increases, $\therefore$ ρ is $+\dfrac{ds}{d\psi}$.

From (1), $$\frac{dv^2}{ds}+\frac{2f}{k^2}v^2=2f.$$

Above is linear and I. F. $=e^{2fs/k^2}$.

Multiplying both sides by I. F. and integrating, we get

$$v^2 \cdot e^{2fs/k^2}=\int 2fe^{2fs/k^2}\,ds+A$$

or $$v^2 \cdot e^{2fs/k^2}=k^2e^{2fs/k^2}+A.$$

When $s=0, v=u$; $\therefore$ $A=u^2-k^2$.

$$\therefore \quad v^2=k^2-(u^2-k^2)\,e^{-2fs/k^2}.$$

Hence from (2) on putting the value of v^2, we get

$$\frac{k^2}{f}+\frac{u^2-k^2}{f}\,e^{-2fs/k^2}=\frac{ds}{d\psi}$$

or

$$e^{2fs/k^2}\frac{ds}{d\psi}-\frac{k^2}{f}\,e^{2fs/k^2}=\frac{u^2-k^2}{f}.$$

Put $\frac{k^2}{2f}\,e^{2fs/k^2}=z$; $\therefore$ $e^{2fs/k^2}\cdot\frac{ds}{d\psi}=\frac{dz}{d\psi}$.

$$\therefore \quad \frac{dz}{d\psi}-2z=\frac{u^2-k^2}{f}.$$

Above is linear and I. F.$=e^{-2\psi}$.

Multiplying both sides by I. F. and integrating, we get

$$z\,.\,e^{-2\psi}=\int\frac{u^2-k^2}{f}\,e^{-2\psi}\,d\psi+B$$

or

$$\frac{k^2}{2f}\,e^{2fs/k^2}\,.\,e^{-2\psi}=\frac{u^2-k^2}{-2f}\,e^{-2\psi}+B.$$

Initially when $s=0$, $\psi=0$.

$$\therefore \quad \frac{k^2}{2f}=\frac{-u^2}{2f}+\frac{k^2}{2f}+B\;;\quad \therefore \quad B=\frac{u^2}{2f}.$$

$$\therefore \quad \frac{k^2}{2f}\,e^{2fs/k^2}\,.\,e^{-2\psi}=\frac{u^2-k^2}{-2f}e^{-2\psi}+\frac{u^2}{2f}$$

or

$$k^2e^{2fs/k^2}=-u^2+k^2+u^2e^{2\psi}.$$

or

$$k^2\,(e^{2fs/k^2}-1)=u^2\,(e^{2\psi}-1).$$

Above is the required intrinsic equation.

Ex. 4. *A particle moving in a resisting medium is acted upon by a central force μ/r^n; if the path be an equiangular spiral of angle α, whose pole is at the centre of force, show that the resistance is* $\frac{n-3}{2}\cdot\frac{\mu\cos}{r^n}$. (Agra 47, 51, 55, 60)

We know that the pedal equation of equiangular spiral of angle α is $p=r\sin\alpha$, so that $\phi=\alpha$ as $p=r\sin\phi$ in any curve.

Also $p=r.\frac{dr}{dp}=r.\frac{1}{\sin\alpha}=r\ \text{cosec}\ \alpha.$

Equations of motion along the tangent and normal are

$$v\frac{dv}{ds}=-\frac{\mu}{r^n}\cos\phi-R \qquad ...(1)$$

and
$$\frac{v^2}{\rho}=\frac{\mu}{r^n}\sin\phi \quad \text{or} \quad v^2=\frac{\mu}{r^n}\sin\phi.\rho \qquad ...(2)$$

or
$$v^2=\frac{\mu}{r^n}\sin\alpha.\ \frac{r}{\sin\alpha}=\frac{\mu}{r^{n-1}}.$$

Differentiating w. r. t. s, we get

$$2v\frac{dv}{ds}=-\frac{(n-1)\mu}{r^n}.\frac{dr}{ds}.$$

But
$$\frac{dr}{ds}=\cos\phi=\cos\alpha.$$

$$\therefore \quad v\frac{dv}{ds}=\frac{-(n-1)\mu}{2r^n}.\cos\alpha.$$

Putting for $v\frac{dv}{ds}$ in (1) and $\phi=\alpha$, we get

$$\frac{-(n-1)\mu}{2r^n}\cos\alpha=\frac{-\mu}{r^n}\cos\alpha-R.$$

$$\therefore \quad R=\frac{\mu}{r^n}\left[\frac{(n-1)}{2}-1\right]\cos\alpha=\frac{\mu}{r^n}.\frac{n-3}{2}.\cos\alpha. \quad \textbf{Proved.}$$

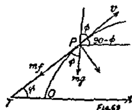
Ex. 5. *A heavy particle is projected in a resisting medium. If v be the velocity at any time t and ϕ the inclination to the vertical of the direction of motion and f the retardation, prove that*

$$\frac{1}{v}\frac{dv}{d\phi}+\cot\phi+\frac{f}{g\sin\phi}=0. \quad \text{(Punjab 56; Raj.}$$

Equations of motion along the tangent and normal are

$$mv\frac{dv}{ds}=-mg\cos\phi-mf$$

and $$m\frac{v^2}{\rho}=mg\sin\phi.$$



Dividing the two equations, we get

$$v\frac{dv}{ds}\cdot\frac{\rho}{v^2}=-\frac{g\cos\phi+f}{g\sin\phi}.$$

or $$\frac{1}{v}\frac{dv}{ds}\left(-\frac{ds}{d\psi}\right)=-\cot\phi-\frac{f}{g\sin\phi}$$

(ψ decreases as s increases)

or $$-\frac{1}{v}\frac{dv}{d\psi}+\cot\phi+\frac{f}{g\sin\phi}=0.$$

or $$-\frac{1}{v}\frac{dv}{d\phi}\cdot\frac{d\phi}{d\psi}+\cot\phi+\frac{f}{g\sin\phi}=0.$$

But $\phi=90-\psi$; $\therefore$ $\frac{d\phi}{d\psi}=-1.$

$$\therefore\ \frac{1}{v}\frac{dv}{d\phi}+\cot\phi+\frac{f}{g\sin\phi}=0.$$ Proved.

Ex. 6. *A particle moves in a resisting medium with a given central acceleration P, the path of the particle being given ; show that the resistance is*

$$-\frac{1}{2p^2}\frac{d}{ds}\left(p^3\frac{dr}{dp}.P\right).$$

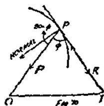
(Punjab 57, 59; Indore 66; Rajputana 65; Agra 48, 62)

The tangential and normal equations of motion are

$$v\frac{dv}{ds}=-P\cos\phi-R, \quad ...(1)$$

and $$\frac{v^2}{\rho}=P\sin\phi=P.\frac{p}{r}, \quad ...(2)$$

$\because\ p=r\sin\phi.$



$$\therefore \quad v^2 = P \cdot \frac{p}{r}\rho = P \cdot \frac{p}{r} \cdot r\frac{dr}{dp} = P \cdot p \cdot \frac{dr}{dp}. \quad \ldots(3)$$

Again from (1), $R = -P\cos\phi - \frac{1}{2}\frac{dv^2}{ds}$.

Putting $\cos\phi = \frac{dr}{ds}$ and v^2 from (3), we get

$$R = -P \cdot \frac{dr}{ds} - \frac{1}{2}\frac{d}{ds}\left(P \cdot p \cdot \frac{dr}{dp}\right)$$

or

$$R = -\frac{1}{2p^2}\left[2P \cdot p^2 \frac{dr}{ds} + p^2 \cdot \frac{d}{ds}\left(P \cdot p \cdot \frac{dr}{dp}\right)\right].$$

We shall make the expression within brackets the differential coefficient of $(p^2)\left(P \cdot p \cdot \frac{dr}{dp}\right)$ w. r. t. s and as such we re-write it as under :

$$R = -\frac{1}{2p^2}\left[P \cdot p \cdot \frac{dr}{dp}\left(2p \cdot \frac{dp}{ds}\right) + p^2 \cdot \frac{d}{ds}\left(P \cdot p \cdot \frac{dr}{dp}\right)\right]$$

or

$$R = -\frac{1}{2p^2}\left[P \cdot p \cdot \frac{dr}{dp} \cdot \frac{d}{ds}(p^2) + p^2 \cdot \frac{d}{ds}\left(P \cdot p \cdot \frac{dr}{dp}\right)\right]$$

or

$$R = -\frac{1}{2p^2} \cdot \frac{d}{ds}\left(p^2 \cdot P \cdot p \cdot \frac{dr}{dp}\right)$$

or

$$R = -\frac{1}{2p^2} \cdot \frac{d}{ds}\left(p^3 \frac{dr}{dp} P\right).$$

Proved.

Alternative Method.

Normal acceleration is $\frac{v^2}{\rho}$ and transverse acceleration is $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$.



Hence the equation of motion are

$$\frac{v^2}{\rho} = P\sin\phi \quad \text{or} \quad \frac{v^2}{r \cdot (dr/dp)} = P \cdot \frac{p}{r}$$

or $$v^2=p.P\frac{dr}{dp}, \quad ...(1)$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=-R\sin\phi.$$

Let $$r^2\frac{d\theta}{dt}=h=v.p. \quad ...(2)$$

$$\therefore \ \frac{1}{r}\frac{dh}{dt}=-R.\frac{p}{r}; \ \therefore \ R=-\frac{1}{p}\frac{dh}{dt}$$

or $$R=-\frac{1}{p}\frac{dh}{ds}.\frac{ds}{dt}=-\frac{v}{p}.\frac{dh}{ds}=-\frac{h}{p^2}\frac{dh}{ds}$$

or $$R=-\frac{1}{2p^2}\frac{d}{ds}(h^2)=-\frac{1}{2p^2}\frac{d}{ds}(v^2p^2) \text{ by (2)}$$

or $$R=-\frac{1}{2p^2}\frac{d}{ds}\left(p.\ P.\frac{dr}{dp}.p^2\right) \text{ by (1)}$$

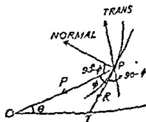
or $$R=-\frac{1}{2p^2}\frac{d}{ds}\left(p^3\frac{dr}{dp}.P.\right).$$

Ex. 7. *If a body moves under a central force P in a medium which exerts a resistance equal to k times the velocity per unit of mass, prove that*

$$u+\frac{d^2u}{d\theta^2}=\frac{P}{h^2u^2}e^{2kt},$$

where h is the initial moment of momentum about the centre of force. **(Vikram 65, Osmania 66, Agra 48)**

Just as in Ex. 6 alternative method, we shall resolve normally and along the transverse.



$$\frac{v^2}{\rho}=P\sin\phi \quad ...(1)$$

and $$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=-kv\sin\phi \quad ...(2)$$

Let us put $r^2\dfrac{d\theta}{dt}=vp=H$ say $=$ moment of momentum.

$$\therefore \quad \frac{1}{r}\frac{dH}{dt}=-k.\frac{H}{p}.\frac{p}{r} \text{ by (2)}$$

or $$\frac{dH}{dt}=-kH \quad \text{or} \quad \frac{dH}{H}=-k\,dt.$$

$$\therefore \quad \log H=-kt+A..$$

Initially $H=h$ (given), $t=0$; $\therefore \quad A=\log h$.

$$\therefore \quad \log\frac{H}{h}=-kt \quad \text{or} \quad H=he^{-kt}. \qquad ...(3)$$

Now from (1), $P=\dfrac{v^2}{\rho \sin\phi}=\dfrac{H^2}{p^2}\cdot\dfrac{1}{r\,.\,dr/dp}\cdot\dfrac{1}{p/r}$

$$P=\frac{H^2}{p^3}\frac{dp}{dr}. \qquad ...(4)$$

Now we know that $\dfrac{1}{p^2}=u^2+\left(\dfrac{du}{d\theta}\right)^2$ where $u=\dfrac{1}{r}$.

$$\therefore \quad -\frac{2}{p^3}\frac{dp}{d\theta}=\left(2u\frac{du}{d\theta}+2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2}\right)$$

or $$-\frac{1}{p^3}\cdot\frac{dp}{d\theta}=\frac{du}{d\theta}\left(u+\frac{d^2u}{d\theta^2}\right)$$

or $$-\frac{1}{p^3}\cdot\frac{dp}{dr}=\frac{du}{dr}\left(u+\frac{d^2u}{d\theta^2}\right)$$

or $$-\frac{1}{p^3}\frac{dp}{dr}=-\frac{1}{r^2}\left(u+\frac{d^2u}{d\theta^2}\right), \quad \because \quad u=\frac{1}{r}.$$

$$\therefore \quad \frac{1}{p^3}\frac{dp}{dr}=u^2\left(u+\frac{d^2u}{d\theta^2}\right). \qquad ...(5)$$

Hence from (4) and (5), we get

$$P=H^2.u^2\left(u+\frac{d^2u}{d\theta^2}\right)=h^2e^{-2kt}.u^2\left(u+\frac{d^2u}{d\theta^2}\right) \text{ by (3)}$$

or $$\frac{P}{h^2u^2}e^{2kt}=u+\frac{d^2u}{d\theta^2}.$$

Proved.

Ex. 8. *A particle moves with central acceleration P in a medium of which the resistance is k (velocity)². Show that the equation to its path is*

$$\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}\,e^{2ks},$$

where s is the length of the arc described and h is the initial moment of momentum about the centre of force. (Agra 50)

It is exactly as Q. 7 with a little difference.

With the same figure, we have

$$\frac{v^2}{\rho}=P \sin \phi, \quad \ldots(1)$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=-kv^2 \sin \phi.$$

Let $r^2\frac{d\theta}{dt}=vp=H$ say$=$moment of momentum.

$$\therefore \quad \frac{1}{r}\frac{dH}{dt}=-k.\frac{H}{p}.\frac{ds}{dt}.\frac{p}{r}.$$

$$\frac{dH}{H}=-k\, ds.$$

Integrating, $\log H=-ks+A$.

Initially when $s=0$, $H=h$; $\therefore$ $A=\log h$.

$$\therefore \quad \log\frac{H}{h}=-ks \quad \text{or} \quad H=he^{-ks}. \quad \ldots(3)$$

Again from (1), $P=\dfrac{v^2}{\rho \sin \phi}=\dfrac{H^2}{p^2}.\dfrac{1}{r\dfrac{dr}{dp}}.\dfrac{1}{\dfrac{p}{r}}$

or $$P=\frac{H^2}{p^3}.\frac{dp}{dr}=h^2e^{-2ks}.\frac{1}{p^3}\frac{dp}{dr}.$$

But from (5) Q. 7,

$$\frac{1}{p^3}\frac{dp}{dr}=u^2\left(u+\frac{d^2u}{d\theta^2}\right)$$

or $$P=h^2e^{-2ks}.u^2\left(u+\frac{d^2u}{d\theta^2}\right) \text{ from (5) of Q. 7}$$

or $$\frac{P}{h^2u^2}e^{2ks}=u+\frac{d^2u}{d\theta^2}.$$ Proved.

CHAPTER V

HODOGRAPH

§ Definition.

Let v be the velocity of the particle at any point P on its path. Through any fixed point O draw a line OQ which is parallel to the direction of velocity of P and is proportional to the magnitude of the velocity of v; then the locus of the point Q is called the hodograph of the motion of P.

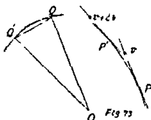


Fig 73

In other words, we can say that radii vectors drawn at a point on the hodograph are parallel to the direction of velocity and proportional to the magnitude of velocity at the corresponding point P on the path of the particle.

Above can also be put as under :—

The co-ordinates of any point Q in the hodograph, i.e. direction of OQ and its magnitude is proportional to the velocity of P in the same direction. ...(1)

Relation between acceleration and velocity.

Let P and P' be the consecutive positions of the particles in its path at times t and $t+\delta t$ where its velocities are v and $v+\delta v$ respectively. Through a fixed point O draw lines OQ and OQ' parallel and proportional to the velocities at points P and P' respectively. Then by definition QQ' is the hodograph of PP'.

Hence change in velocity during the short interval δt from P to P' is represented by chord QQ'

$\therefore$ acceleration at P = rate of change of velocity at P

$$= \underset{\delta t \to 0}{\text{Lt}} \frac{\text{chord } QQ'}{\delta t}$$

or

$$\text{acceleration at } P = \underset{\delta t \to 0}{\text{Lt}} \frac{\text{chord } QQ'}{\text{arc } QQ'} \cdot \frac{\text{arc } QQ'}{\delta t}$$

$$= 1 \,.\, \text{velocity at } Q,$$

$$\because \text{ Lt } \frac{\delta c}{\delta s} = 1 \text{ and Lt } \frac{\delta s}{\delta t} = \frac{ds}{dt} = \text{velocity.}$$

Hence from above we conclude that

Acceleration at any point P on the curve is equal *to the* velocity at the corresponding point Q on the hodograph. ...(2)

Working rule. Cartesian equation.

Let $P(x, y)$ be a point on the path of the particle and $Q(X, Y)$ be a corresponding point on the hodograph; then by result (1), *i.e.* co-ordinates of a point on the hodograph are parallel and proportional to the velocity of the corresponding point on the path of the particle, we have

$$X = \lambda \frac{dx}{dt} \text{ and } Y = \lambda \frac{dy}{dt} \,. \qquad ...(3)$$

Also by result (2), *i.e.* velocity at any point Q on the hodograph is equal to the acceleration at the corresponding point P on the path of the particle, we have

$$\frac{dX}{dt} = \lambda \frac{d^2x}{dt^2} \text{ and } \frac{dY}{dt} = \lambda \frac{d^2y}{dt^2} \,.$$

Working rule in polar equation.

We know that if (r, θ) be any point in polar and corresponding cartesian co-ordinates by (X, Y), then

$$\sqrt{(X^2+Y^2)} = r \text{ and } \tan \theta = \frac{Y}{X}.$$

Also if v be the velocity at any point (x, y), then

$$v=\sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]} \text{ and } \tan\psi=\frac{dy}{dx}$$

where ψ is the angle which the direction of velocity makes with a fixed direction.

Hence from (3), we get

$$\sqrt{(X^2+Y^2)}=\lambda\sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]}$$

or $$r=\lambda v$$

and $$\frac{Y}{X}=\frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}=\frac{dy}{dx} \quad \text{or} \quad \tan\theta=\tan\psi\ ; \quad \therefore \quad \theta=\psi.$$

Hence to find the hodograph *i.e.* locus of (X, Y) or (r, θ), any point on the hodograph, we should find a relation between v and ψ, *i.e.* $v=f(\psi)$ and in it replace v by $\frac{r}{\lambda}$ and ψ by θ so that the polar equation of the hodograph is given by

$$\frac{r}{\lambda}=f(\theta) \quad \text{or} \quad r=\lambda f(\theta).$$

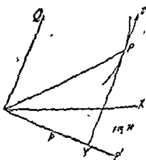
The above is evident otherwise too, *i.e.* $OQ=r$ is proportional to the magnitude of velocity, $r=\lambda v$. Also OQ is parallel to the direction of velocity ; therefore $\theta=\psi$.

Note. If the equation of the hodograph be given, then we should replace r by λv and θ by ψ which will give us v and ψ relation from which we shall find the equation of the path by the help of other additional given condition.

§ 2. Hodograph of a central orbit.

The hodograph of a central orbit is the reciprocal of the orbit with respect to the centre of force S turned through a right angle about S. **(Indore 66 ; Raj. 62 ; Agra 46, 58, 60, 63)**

Definition of reciprocal.



Let p be the perpendicular SY from S on the tangent at P to a given curve. Produce SY to P' such that $SY.SP'=\text{constant}=k^2$ say; then the locus of P' is called the reciprocal (polar reciprocal) of the locus of P *i.e.* the path of P.

Now in a central orbit, we know that

$$v.p=h$$

or

$$v.SY=h \quad \text{or} \quad v.\frac{k^2}{SP'}=h$$

or

$$SP'=\frac{k^2}{h}.v. \qquad ..(1)$$

Above shows that SP' is proportional to the velocity at P but perpendicular to its direction. But by definition we know that hodograph is the locus of a point Q such that SQ is proportional and parallel to the velocity of P.

Hence we can say that the hodograph of P is the locus of P' turned through a right angle ($\angle P'SQ=90°$) about S. But locus of P' where $SY.SP'=\text{constant}$ is by definition the reciprocal of the orbit. Therefore the hodograph of a central orbit is the reciprocal of the orbit with respect to centre of force but turned through a right angle.

From (1), $SP'=\frac{k^2}{h}.\text{velocity at } P.$

$\therefore$ velocity at $P'=\frac{k^2}{h}.\text{acceleration at } P$

or velocity at P' is proportional to central acceleration.

Working rule. Take (r_1, θ_1) the foot Y of the perpendicular from centre of force and find a relation between (r_1, θ_1), since $SY.SP'=k^2$ or $r_1.SP'=k^2$ where P' is on SY produced

(vectorial angle of P is same as of Y). Replace r_1 by k^2/r_1 and generalize. Thus you will get the locus of P' *i.e.* reciprocal of given orbit. Then turn it through a right angle to get the hodograph.

§ 3. Hodograph of central orbit. Pedal equation.

(Agra 62)

Let the pedal equation of the central orbit be $p=f(r)$ and any point P on it be (p, r). Suppose that the co-ordinates of P' be (p', r') and those of Y be (p_1, r_1) in figure of § 2.

Now $\quad p=SY=r_1$; $\because$ Y is (p_1, r_1). ...(1)

Also we know that $\quad SY.SP'=\text{constant}=k^2$ as in § 2.

or $$r_1 . r'=k^2. \qquad ...(2)$$

Again if $\angle SPY=\phi$, then

$$\sin\phi=\frac{SY}{SP}=\frac{p}{r}=\frac{p_1}{r_1}=\frac{p'}{r'}.$$

$\therefore\ p'r=pr'$ or $p'r=r_1 r'$ by (1)

or $\quad p'r=k^2$ by (2).

$$\therefore\ p'=\frac{k^2}{r} \text{ and } r'=\frac{k^2}{r_1}=\frac{k^2}{p} \text{ by (1)}.$$

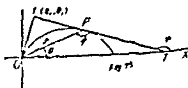
Thus we have $\quad p'=\dfrac{k^2}{r}\quad$ and $r'=\dfrac{k^2}{p}$. ...(3)

Hence the hodograph which is polar reciprocal turned through a right angle is obtained by using relation (3) and then turning through a right angle.

Hence the hodograph of $p=f(r)$ is the locus of (p', r') and is $\dfrac{k^2}{r}=f\left(\dfrac{k^2}{p}\right)$ turned through a right angle.

Ex. 1. *If a particle describes a lemniscate under a force to its pole, show that the equation to the hodograph is*

$$r^2=a^2 \sec^3\left(\frac{\pi-2\theta}{3}\right). \qquad \text{(Agra 45, 49, 52, 65)}$$



Let the equation of the lemniscate be $r^2=a^2 \cos 2\theta$. Taking log of both sides, we get

$2 \log r=2 \log a+\log \cos 2\theta$.

Differentiating with respect to θ, we get

$$\frac{2}{r}\frac{dr}{d\theta}=\frac{1}{\cos 2\theta}(-2 \sin 2\theta)$$

or $$\cot \phi=-\tan 2\theta=\cot\left(\frac{\pi}{2}+2\theta\right).$$

$$\therefore \quad \phi=\frac{\pi}{2}+2\theta \quad \text{or} \quad \psi=\theta+\phi=\frac{\pi}{2}+3\theta. \quad ...(1)$$

We know that $p=r \sin \phi=r \cos 2\theta=a (\cos 2\theta)^{3/2}$

or $$p^2=a^2 \cos^3 2\theta=a^2 \cos^3 \frac{2}{3}\left(\psi-\frac{\pi}{2}\right) \text{ by (1)}. \quad ...(2)$$

In a central orbit, $vp=h$. $\therefore \quad p^2=\dfrac{h^2}{v^2}$.

Hence from (2), $\dfrac{h^2}{v^2}=a^2 \cos^3\left(\dfrac{2\psi-\pi}{3}\right)$.

Above is a v, ψ relation of the path and in order to obtain its hodograph replace v by r/λ and ψ by θ (§ 1, working rule).

$\therefore$ Hodograph is $\dfrac{h^2}{r^2}\lambda^2=a^2 \cos^3\left(\dfrac{2\theta-\pi}{3}\right)$

or $$r^2=\frac{h^2\lambda^2}{a^2} \sec^3\left(\frac{\pi-2\theta}{3}\right)$$

or $$r^2=A^2 \sec^3\left(\frac{\pi-2\theta}{3}\right).$$

We may take a^2 for A^2 to get the answer in the given form.

Alternative Method.

Let Y be the foot of perpendicular from centre of force

Differentiating,

$$\frac{1}{r}\frac{dr}{d\theta}=\cot\alpha \quad \text{or} \quad \cot\phi=\cot\alpha\,; \quad \therefore \quad \phi=\alpha.$$

Also $$\psi=\theta+\phi=\theta+\alpha.$$

$$\therefore \quad p=r\sin\phi=r\sin\alpha$$

or $$p=ae^{\theta\cot\alpha}.\sin\alpha=a\sin\alpha.e^{(\psi-\alpha)\cot\alpha}.$$

In a central orbit $v.p=h$; $\therefore \quad p=\frac{h}{v}$.

$$\therefore \quad \frac{h}{v}=a\sin\alpha.e^{-\alpha\cot\alpha}e^{\psi\cot\alpha}.$$

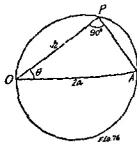
Above is a v, ψ relation. From § 1, we know that the hodograph is obtained by replacing ψ by θ and v by r/λ. Hence hodograph is

$$\frac{h}{r/\lambda}=a\sin\alpha.e^{-\alpha\cot\alpha}e^{\theta\cot\alpha}$$

or $$r=\frac{\lambda h}{a\sin\alpha.e^{-\alpha\cot\alpha}}.e^{-\theta\cot\alpha}$$

or $r=be^{-\theta\cot\alpha}$ which is also an equiangular spiral.

Ex. 3. *If the hodograph be a circle described with constant angular velocity about a point on its circumference, show that the path is a cycloid.* (Agra 55)



Here is a reverse problem; we are given the hodograph, and we are to find the path.

The equation of the hodograph, *i. e.* a circle referred to a point O on it as pole and the diameter through O as initial line is

$$r=2a\cos\theta. \qquad \text{...(1)}$$

We know from § 1 that if (r, θ) be the coordinates of a point on the hodograph, then $r=\lambda v$ and $\theta=\psi$ where v

represents the velocity and ψ the direction of velocity. Hence from (1), $\lambda v=2a\cos\psi$. ...(2)

We are given that the circle is described with constant angular velocity.

$$\therefore \quad \frac{d\theta}{dt}=\omega \quad \text{or} \quad \frac{d\psi}{dt}=\omega \qquad \because \quad \psi=\theta.$$

From (2), $\lambda.\dfrac{ds}{dt}=2a\cos\psi$ or $\lambda.\dfrac{ds}{d\psi}.\dfrac{d\psi}{dt}=2a\cos\psi$

or $\qquad \lambda.\dfrac{ds}{d\psi}\,\omega=2a\cos\psi$ or $\dfrac{\lambda\omega}{2a}\,ds=\cos\psi\,d\psi.$

Integrating, we get $s=\dfrac{2a}{\lambda\omega}\sin\psi+B.$

Let us suppose that when $s=0,\ \psi=0,\ \therefore\ B=0.$

Hence $s=\dfrac{2a}{\lambda\omega}\sin\psi$ or $s=4b\sin\psi$ is the equation of the path which we know is the intrinsic equation of the cycloid $[s=4a\sin\psi]$.

Ex. 4. *Show that the hodograph of a circle under a force to a point on the circumference is a parabola.*

(Vikram 65 ; Agra 59)

Just as in Q. 3 the equation of the path is $r=2a\cos\theta$ and we are to find its hodograph.

$$\frac{dr}{d\theta}=-2a\sin\theta\ ; \quad \therefore \quad \tan\phi=r\frac{d\theta}{dr}=\frac{2a\cos\theta}{-2a\sin\theta}=-\cot\theta.$$

$$\therefore \quad \phi=\frac{\pi}{2}+\theta \quad \text{or} \quad \psi=\theta+\phi=\frac{\pi}{2}+2\theta \text{ or } 2\theta=\left(\psi-\frac{\pi}{2}\right).$$

Again $p=r\sin\phi=r\cos\theta=r.\dfrac{r}{2a}=\dfrac{4a^2\cos^2\theta}{2a},$

$$p=a\,(1+\cos 2\theta).$$

But in a central orbit $v.p=h$ or $p=\dfrac{h}{v}.$

Also $2\theta=\psi-\dfrac{\pi}{2}\ ;\ \therefore\ \dfrac{h}{v}=a\left\{1+\cos\left(\psi-\dfrac{\pi}{2}\right)\right\}=a\,(1+\sin\psi).$

Above is v, ψ relation and in order to find hodograph replace ψ by θ and v by $\frac{r}{\lambda}$ [§ 1].

$$\therefore \quad \frac{h.\lambda}{r}=a\,(1+\sin\theta).$$

$$\therefore \quad \text{Hodograph is } \frac{2b}{r}=1+\sin\theta \text{ where } 2b=\frac{h\lambda}{a}.$$

Above equation represents a parabola. (polar equation)

Ex. 5. (a) *If the path be an ellipse described under a force to its centre, show that the hodograph is a similar ellipse.*

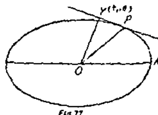


Fig 77

Since the force is to the centre of ellipse, so we consider the cartesian equation and not the polar which is referred to focus as pole.

Let Y be the foot of perpendicular from O on the tangent at P and its co-ordinates be (r_1, θ_1) i.e. $OY=r_1$ and $\angle YOA=\theta_1$, so that the equation of tangent is

$\therefore$ Hodograph is $a^2 \sin^2 \theta + b^2 \cos \theta = \frac{k^4}{r^2}$

or $$k^4 = a^2 (r^2 \sin^2 \theta) + b^2 (r^2 \cos^2 \theta)$$

or $$1 = \frac{x^2}{k^4/b^2} + \frac{y^2}{k^4/a^2}$$ which is a similar ellipse.

(b) *A particle describes a conic section under a force to its focus ; show that the hodograph is a circle which passes through the centre of force when the path is a parabola.*

Refer figure part (a) with the difference that instead of centre of force at O, we have to take it at focus S which may be taken as pole, so that the polar equation of the conic is

$$l/r = 1 + e \cos \theta.$$

Let the vectorial angle of P be α, so that the tangent at α to above is

$$\frac{l}{r} = \cos (\theta - \alpha) + e \cos \theta \quad ...(1)$$

or $$l = r \cos \theta (\cos \alpha + e) + r \sin \theta \sin \alpha$$

or $$l = x (\cos \alpha + e) + y \sin \alpha.$$

Again SY is perpendicular from focus, *i.e.* pole on it, so that its equation is

$$0 = x \sin \alpha - y (\cos \alpha + e)$$

or $$0 = r [\cos \theta \sin \alpha - \sin \theta (\cos \alpha + e)]$$

or $$\sin (\theta - \alpha) = -e \sin \theta \quad ...(3)$$

Let the foot of perpendicular from S, *i.e.* point Y be (r, θ_1) which lies on both (1) and (3).

$$\therefore \quad \frac{l}{r_1} - e \cos \theta_1 = \cos (\theta_1 - \alpha)$$

and $$-e \sin \theta_1 = \sin (\theta_1 - \alpha).$$

Eliminating variable r, we get

$$\left(\frac{1}{r_1} - e \cos \theta_1\right)^2 + e^2 \sin^2 \theta_1 = 1. \quad ..(4)$$

Above is a relation between (r_1, θ_1) the co-ordinates of

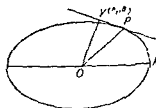
Above is r, ψ relation and in order to find hodograph replace ψ by θ and r by $\frac{r}{\lambda}$ [§ 1].

$$\therefore \quad \frac{h.\lambda}{r}=a\,(1+\sin\,\theta).$$

$\therefore$ Hodograph is $\frac{2b}{r}=1+\sin\,\theta$ where $2b=\frac{h\lambda}{a}$.

Above equation represents a parabola. (polar equation)

Ex. 5. (a) *If the path be an ellipse described under a force to its centre, show that the hodograph is a similar ellipse.*



Since the force is to the centre of ellipse, so we consider the cartesian equation and not the polar which is referred to focus as pole.

Let Y be the foot of perpendicular from O on the tangent at P and its co-ordinates be (r_1, θ_1) *i e.* $OY=r_1$ and $\angle YOA=\theta_1$, so that the equation of tangent is

$$x\cos\,\theta_1+y\sin\,\theta_1=r_1 \qquad [x\cos\,\alpha+y\sin\,\alpha=p].$$

Since it is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$\therefore$ Condition of tangency gives $a^2l^2+b^2m^2=n^2$,

i e. $$a^2\cos^2\,\theta_1+b^2\sin^2\,\theta_1=r_1^2. \qquad \ldots(1)$$

The reciprocal of the given ellipse is the locus of P' on OY produced such that $OY.OP'=k^2$.

Hence in (1) replacing r_1 by $\frac{k^2}{r_1}$ and generalizing the locus of P' is $a^2\cos^2\theta+b^2\sin^2\theta=\frac{k^4}{r^2}$ which is the reciprocal of given ellipse. Hodograph is the reciprocal of given curve turned through a right angle.

$\therefore$ Hodograph is $a^2 \sin^2 \theta + b^2 \cos \theta = \frac{k^4}{r^2}$

or $\qquad k^4 = a^2 (r^2 \sin^2 \theta) + b^2 (r^2 \cos^2 \theta)$

or $\qquad 1 = \frac{x^2}{k^4/b^2} + \frac{y^2}{k^4/a^2}$ which is a similar ellipse.

(b) *A particle describes a conic section under a force to its focus ; show that the hodograph is a circle which passes through the centre of force when the path is a parabola.*

Refer figure part (a) with the difference that instead of centre of force at O, we have to take it at focus S which may be taken as pole, so that the polar equation of the conic is

$$l/r = 1 + e \cos \theta.$$

Let the vectorial angle of P be α, so that the tangent at α to above is

$$\frac{l}{r} = \cos (\theta - \alpha) + e \cos \theta \qquad \ldots(1)$$

or $\qquad l = r \cos \theta (\cos \alpha + e) + r \sin \theta \sin \alpha$

or $\qquad l = x (\cos \alpha + e) + y \sin \alpha.$

Again SY is perpendicular from focus, *i.e.* pole on it, so that its equation is

$$0 = x \sin \alpha - y (\cos \alpha + e)$$

or $\qquad 0 = r [\cos \theta \sin \alpha - \sin \theta (\cos \alpha + e)]$

or $\qquad \sin (\theta - \alpha) = -e \sin \theta \qquad \ldots(3)$

Let the foot of perpendicular from S, *i.e.* point Y be (r, θ_1) which lies on both (1) and (3).

$$\therefore \quad \frac{l}{r_1} - e \cos \theta_1 = \cos (\theta_1 - \alpha)$$

and $\qquad -e \sin \theta_1 = \sin (\theta_1 - \alpha).$

Eliminating variable α, we get

$$\left(\frac{l}{r_1} - e \cos \theta_1\right)^2 + e^2 \sin^2 \theta_1 = 1. \qquad \ldots(4)$$

Above is a relation between (r_1, θ_1) the co-ordinates of

Y. The reciprocal of given conic is the locus of P' on SY produced such that

$$SY\ SP'=k^2 \quad \text{or} \quad SP'=\frac{k^2}{SY}=\frac{k^2}{r_1}.$$

The vectorial angle of P' will be same as that of Y, *i.e.* θ_1.

Hence replacing r_1 by $\frac{k^2}{r_1}$ in (4), we get

$$\left(\frac{lr_1}{k^2}-e\cos\theta_1\right)^2+e^2\sin^2\theta_1=1.$$

Generalising, we get the reciprocal as

$$\left(\frac{lr}{k^2}-e\cos\theta\right)^2+e^2\sin^2\theta=1.$$

Hodograph is the reciprocal turned through a right angle. Hence the equation of the hodograph is

$$\left[\frac{lr}{k^2}-e\cos\left(\frac{\pi}{2}+\theta\right)\right]^2+e^2\sin^2\left(\frac{\pi}{2}+\theta\right)=1$$

or
$$\left(\frac{lr}{k^2}+e\sin\theta\right)^2+e^2\cos^2\theta=1$$

or
$$\frac{l^2}{k^4}r^2+\frac{2le}{k^2}r\sin\theta+e^2(\cos^2\theta+\sin\theta)=1$$

or
$$l^2(x^2+y^2)+2lek^2.y+k^4(e^2-1)=0.$$

Above is a circle.

If the conic be a parabola, *i.e.* $e=1$, then hodograph is

$$l^2(x^2+y^2)-2lk^2y=0 \quad \text{or} \quad x^2+y^2-\frac{2k^2}{l}y=0,$$

which is a circle passing through origin, *i.e.* pole or focus, which is centre of force in our equation.

(c) *A circle is described under a force to an internal point. Show that the hodograph of the motion is an ellipse.*

(Agra 59)

Let the centre of force be the point O such that $OC=c$ and Y be the foot of perpendicular from O on tangent at P and its co-ordinates be (r_1, θ_1); then

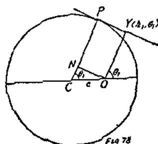


Fig 78

$$r_1=OY=PN=CP-CN$$

$$=a-c\cos\theta_1. \qquad ...(1)$$

$\therefore$ The reciprocal of this circle is the locus of P' on OY produced such that $OY.OP'=k^2$. Hence in (1) replacing r_1 by $\frac{k^2}{r_1}$ and generalizing, we get the locus of P' as $\frac{k^2}{r}=a-c\cos\theta$ which is reciprocal of given circle. We know that hodograph is the reciprocal of the given curve turned through a right angle.

$$\therefore \quad \text{hodograph is } \frac{k^2}{r}=a-c\cos\left(\frac{\pi}{2}+\theta\right)$$

or $\frac{k^2/a}{r}=1+\frac{c}{a}\sin\theta$ which is of the form $\frac{l}{r}=1+e\sin\theta$ which

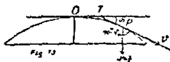
represents an ellipse as the eccentricity $e=\frac{c}{a}$ is less than 1,

$$\because c<a.$$

Ex. 6. *A particle slides down in a thin cycloidal tube, whose axis is vertical and vertex the highest point. Show that the equation to the hodograph is of the form $r^2=2g\,(a+b\cos 2\theta)$ the particle starting from any point of the cycloid.*

If it starts from the highest point, show that the hodograph is a circle. (Agra 57)

Here we shall find v and ψ relation where v is the velocity at any point P and ψ is the direction of motion.



Tangential equation of motion is

$$mv\frac{dv}{ds}=mg\sin\psi \quad \text{or} \quad v\frac{dv}{ds}=g\frac{s}{4a}.$$

Because the intrinsic equation of a cycloid is $s=4a\sin\psi$, integrating, we get $v^2=\frac{g}{4a}s^2+A$.

Let us suppose that the particle started where $s=s_0$ so that when $v=0$, $s=s_0$. $\therefore$ $A=-\frac{g}{4a}s_0^2$.

$$\therefore \quad v^2=\frac{g}{4a}(s^2-s_0^2)$$

or
$$v^2=\frac{g}{4a}(16a^2\sin^2\psi-s_0^2). \quad ..(1)$$

Above is the v, ψ relation and in order to find the hodograph, we put $\psi=\theta$ and $r=v\lambda$ *i.e.* $v=\frac{r}{\lambda}$.

$\therefore$ Hodograph is $\frac{r^2}{\lambda^2}=\frac{g}{4a}(16a^2\sin^2\theta-s_0^2)$

or
$$r^2=\frac{g\lambda^2}{4a}\{8a^2(1-\cos 2\theta)-s_0^2\}. \quad ...(2)$$

Above can be put into the form $r^2=2g(a+b\cos 2\theta)$.

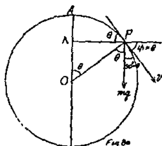
In case the particle started from the highest point, then $v=0$ when $s_0=0$ (s being measured from the vertex O). Hence putting $s_0=0$ in (2), the equation of the hodograph is

$$r^2=\frac{g\lambda^2}{4a}\,8a^2.(1-\cos 2\theta)=\frac{g\lambda^2}{4a}.8a^2.2\sin^2\theta$$

or $\quad r^2=d^2\sin^2\theta$ or $r=d\sin\theta$ or $r=d\cos(\theta-90)$ which represents a circle whose diameter of length d is inclined at $90°$ to the initial line.

Ex. 7. *A bead moves on the arc of a smooth vertical circle starting from rest at the highest point. Show that the*

equation to the hodograph is $r=\lambda \sin\frac{\theta}{2}$. (Agra 59)



Let P be the position of the particle. Tangential equation of motion is

$$mv\frac{dv}{ds}=mg\sin\theta$$

or $$v\,dv=g\sin\theta\cdot\frac{ds}{d\theta}\,d\theta.$$

But $s=a\theta$ in a circle; $\therefore\ \frac{ds}{d\theta}=a.$

$$\therefore\quad \frac{v^2}{2}=-ag\cos\theta+A,$$

Initially when $\theta=0$, $v=0$; $A=ag$.

$$\therefore\quad v^2=2ag\,(1-\cos\theta). \quad ...(1)$$

If ψ be the angle which the direction of motion at P makes with the horizontal, then from the figure it is clear that $\psi=\theta$. Hence from (1),

$$v^2=2ag\ (1-\cos\psi)=2ag\,.\,2\sin^2\frac{\psi}{2}.$$

$$\therefore\quad v=2\sqrt{(ag)}\sin\frac{\psi}{2}\ . \quad ...(2)$$

Above is the v, ψ relation and hodograph is obtained by writing θ for ψ and $r=kv$, *i.e.* $v=\frac{r}{k}$.

Hodograph is $\frac{r}{k}=2\sqrt{(ag)}\sin\frac{\theta}{2}$ or $r=2k\sqrt{(ag)}\sin\frac{\theta}{2}$

which is of the form $r=\lambda\sin\frac{\theta}{2}$. Proved.

Note. The equation (1) could be directly written from the energy equation. Change in K. E.=work done by weight in falling a vertical distance AN,

i. e. $\frac{1}{2}m(v^2-0)=mg\,.\,AN=mg\,(OA-ON)=mg\,(a-a\cos\theta)$.

$$\therefore\quad v^2=2ag\,(1-\cos\theta).$$

Ex. 8. *A smooth elliptic tube is placed with its major axis vertical and a particle is allowed to slide down starting from rest at the highest point. Show that the hodograph is*

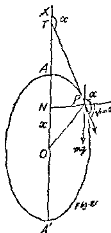
$$r=c\sin\frac{1}{2}\left\{\cot^{-1}\left(\frac{a}{b}\right)\cot\theta\right\},$$

the angle θ being with the horizontal. **(Rajputana 62)**

The angle θ in the equation of the hodograph is the same as ψ which occurs in v, ψ relation in the motion of the particle. Since θ is measured from the horizontal, hence ψ is the angle which the direction of motion makes with the horizontal.

Let the equation of the ellipse be $x=a\cos t$, $y=b\sin t$ so that the co-ordinates of any point P on it are

$(a\cos\phi,\ b\sin\phi)$.



Also we know that in Cartesian co-ordinates $\frac{dy}{dx}$ stands for tangent of the angle which the tangent at any point makes with the + ive direction of the axis of x and if the angle be α, then

$$\frac{dy}{dx}=\tan\alpha$$

or
$$\frac{dy}{dt}\bigg/\frac{dx}{dt}=\tan\alpha,$$

or
$$\frac{b\cos t}{-a\sin t}=\tan\alpha\quad\text{or}\quad-\frac{b}{a}\cot t=\tan\alpha.$$

$$\therefore \quad \cot t = -\frac{a}{b} \tan \alpha$$

or

$$\cot t = -\frac{a}{b} \cot (90-\alpha) = \frac{a}{b} \cot (\alpha - 90°). \qquad \ldots (1)$$

From the figure it is clear that $\psi = \alpha - 90°$.

$$\therefore \quad \cot t = \frac{a}{b} \cot \psi \text{ by (1).} \qquad \ldots (2)$$

From the equation of energy if v be the velocity when the particle is at P,
change in K. E. = work done by the wt. in falling through AN.

$$\therefore \quad \tfrac{1}{2} m (v^2 - 0) = mg. \ AN = mg \ (OA - ON) = mg \ (a - x)$$

or

$$v^2 = 2g \ (a - a \cos t) = 2ga \,.\, 2 \sin^2 \frac{t}{2}.$$

$$\therefore \quad v = 2\sqrt{(ag)} \sin \frac{t}{2}$$

or

$$v = 2\sqrt{(ag)} \sin \frac{1}{2} \left\{ \cot^{-1} \left(\frac{a}{b} \cot \psi \right) \right\} \text{ by (2),}$$

where ψ is measured from the horizontal.

Above is the v and ψ relation and the hodograph is obtained by writing θ for ψ and $r = \lambda v$, *i.e.* $v = \frac{r}{\lambda}$.

$$\therefore \quad \text{Hodograph is } \frac{r}{\lambda} = 2\sqrt{(ag)} \sin \frac{1}{2} \left\{ \cot^{-1} \left(\frac{a}{b} \cot \theta \right) \right\}$$

or

$$r = c \sin \frac{1}{2} \left\{ \cot^{-1} \left(\frac{a}{a} \cot \theta \right) \right\}. \qquad \textbf{Proved.}$$

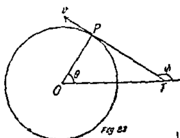
Ex. 9. *Show that the polar equation of the hodograph of a point which describes a circle of radius a with a constant angular acceleration* α *is*

$$r^2 = k^2 a^2 \ [\alpha \ (2\theta - \pi) + \omega^2],$$

k being a constant, the initial angular position and velocity being zero and ω *respectively.*

We are given that

$$\frac{d^2\theta}{dt^2}=\alpha, \text{ constant.}$$



Integrating, we get

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2=\alpha\theta+A.$$

Initially when $\theta=0$, it is given that $\frac{d\theta}{dt}=\omega$; $\therefore$ $A=\frac{1}{2}\omega^2$.

$$\therefore \quad \left(\frac{d\theta}{dt}\right)^2=2a\theta+\omega^2. \qquad ...(1)$$

But in a circle $\quad s=a\theta,\ v=\frac{ds}{dt}=a\frac{d\theta}{dt}$.

Also from the figure $\psi=90+\theta$; $\quad\therefore\quad \theta=\psi-90$.

Hence from (1) $\quad v^2=a^2\ (2\alpha\theta+\omega^2)$

or $\quad v^2=a^2\ [2\alpha\ (\psi-90)+\omega^2]$.

Above is the v, ψ relation and hence the hodograph is obtained by putting θ for ψ and $r=\lambda v$.

$$\therefore \quad r^2=\lambda^2a^2\ [\alpha\ (2\theta-\pi)+\omega^2]. \qquad \textbf{Proved.}$$

Ex. 10. *If P and Q be the tangential and normal forces and ψ the inclination of tangent to a fixed line, the hodograph is*

$$\log\frac{r}{a}=\int_0^{\theta}\frac{P}{Q}\,d\psi.$$

Equations of motion of the particle are

$$mv\frac{dv}{ds}=P \text{ and } m\frac{v^2}{\rho}=Q.$$

Dividing, we get $\frac{1}{v}\frac{dv}{ds}\cdot\rho=\frac{P}{Q}$. Put $\rho=\frac{ds}{d\psi}$.

$$\therefore \quad \frac{1}{v}\frac{dv}{ds}\cdot\frac{ds}{d\psi}=\frac{P}{Q} \quad\text{or}\quad \frac{dv}{v}=\frac{P}{Q}\,d\psi.$$

Integrating, we get $\qquad \log v=\int_0^{\psi}\frac{P}{Q}\,d\psi+A.$

Taking $A=\log c$, we get

$$\log \frac{v}{c}=\int_0^{\psi} \frac{P}{Q}\, d\psi.$$

Above is v, ψ relation of the motion of the particle.

In order to obtain the hodograph replace ψ by θ and $r=\lambda v$.

$$\therefore \quad \log \frac{r}{c\lambda}=\int_0^{\theta} \frac{P}{Q}\, d\theta \quad \text{or} \quad \log \frac{r}{a}=\int_0^{\theta} \frac{P}{Q}\, d\theta. \qquad \textbf{Proved.}$$

Ex. 11. *The hodograph of an orbit is a parabola whose ordinate increases uniformly Show that the orbit is a semi-cubical parabola.* **(Agra 54, 61)**

From working rule § 1, we know that co-ordinates of a point on the hodograph are parallel and proportional to the velocity of the corresponding point on the path of the particle. Then if (X, Y) be a point on the hodograph and corresponding point on the path be (x, y), then

$$X=\lambda \frac{dx}{dt} \text{ and } Y=\lambda \frac{dy}{dt}. \qquad \ldots(1)$$

Since the hodograph is a parabola, let its equation be

$$Y^2=4aX. \qquad \ldots(2)$$

Since its ordinate increases at a uniform rate, $\therefore \ \frac{dY}{dt}=k$ say.

$\therefore \quad Y=kt+A$ if we choose that when $t=0$, $Y=0$, then $A=0$.

$$\therefore \quad Y=kt\ ; \quad \therefore \quad X=\frac{Y^2}{4a}=\frac{k^2}{4a}\, t^2 \text{ from (2).}$$

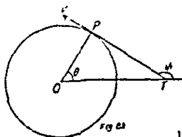
Hence from (1), we get $\frac{k^2}{4a}\, t^2=\lambda \frac{dx}{dt}=X$

and

$$kt=\lambda \frac{dy}{dt}=Y.$$

We are given that

$$\frac{d^2\theta}{dt^2}=\alpha, \text{ constant.}$$



Integrating, we get

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2=\alpha\theta+A.$$

Initially when $\theta=0$, it is given that $\frac{d\theta}{dt}=\omega$; $\therefore$ $A=\frac{1}{2}\omega^2$.

$$\therefore \quad \left(\frac{d\theta}{dt}\right)^2=2\alpha\theta+\omega^2. \qquad \text{...(1)}$$

But in a circle $\quad s=a\theta,\ v=\dfrac{ds}{dt}=a\dfrac{d\theta}{dt}.$

Also from the figure $\psi=90+\theta$; $\therefore$ $\theta=\psi-90$.

Hence from (1) $\quad v^2=a^2\ (2\alpha\theta+\omega^2)$

or $\quad v^2=a^2\ [2\alpha\ (\psi-90)+\omega^2]$.

Above is the v, ψ relation and hence the hodograph is obtained by putting θ for ψ and $r=\lambda v$.

$$\therefore \quad r^2=\lambda^2a^2\ [\alpha\ (2\theta-\pi)+\omega^2]. \qquad \textbf{Proved.}$$

Ex. 10. *If P and Q be the tangential and normal forces and ψ the inclination of tangent to a fixed line, the hodograph is*

$$\log\frac{r}{a}=\int_0^\psi \frac{P}{Q}\,d\psi.$$

Equations of motion of the particle are

$$mv\frac{dv}{ds}=P \text{ and } m\frac{v^2}{\rho}=Q.$$

Dividing, we get $\frac{1}{v}\frac{dv}{ds}\cdot\rho=\frac{P}{Q}$. Put $\rho=\frac{ds}{d\psi}$.

$$\therefore \quad \frac{1}{v}\frac{dv}{ds}\cdot\frac{ds}{d\psi}=\frac{P}{Q} \quad \text{or} \quad \frac{dv}{v}=\frac{P}{Q}\,d\psi.$$

Integrating, we get $\quad \log v=\displaystyle\int_0^\psi \frac{P}{Q}\,d\psi+A.$

Taking $A=\log c$, we get

$$\log \frac{v}{c}=\int_0^{\psi} \frac{P}{Q}\, d\psi.$$

Above is v, ψ relation of the motion of the particle.

In order to obtain the hodograph replace ψ by θ and $r=\lambda v$.

$$\therefore \quad \log \frac{r}{c\lambda}=\int_0^{\theta} \frac{P}{Q}\, d\theta \quad \text{or} \quad \log \frac{r}{a}=\int_0^{\theta} \frac{P}{Q}\, d\theta. \qquad \textbf{Proved.}$$

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From working rule § 1, we know that co-ordinates of a point on the hodograph are parallel and proportional to the velocity of the corresponding point on the path. of the particle. Then if (X, Y) be a point on the hodograph and corresponding point on the path be (x, y), then

$$X=\lambda \frac{dx}{dt} \text{ and } Y=\lambda \frac{dy}{dt}. \qquad \ldots(1)$$

Since the hodograph is a parabola, let its equation be

$$Y^2=4aX. \qquad \ldots(2)$$

Since its ordinate increases at a uniform rate, $\therefore \frac{dY}{dt}=k$ say.

$\therefore \quad Y=kt+A$ if we choose that when $t=0$, $Y=0$, then $A=0$.

$$\therefore \quad Y=kt\,; \quad \therefore \quad X=\frac{Y^2}{4a}=\frac{k^2}{4a}\, t^2 \text{ from (2).}$$

Hence from (1), we get $\frac{k^2}{4a}\, t^2=\lambda \frac{dx}{dt}=X$

and $$kt=\lambda \frac{dy}{dt}=Y.$$

Integrating, we get $x=\frac{k^2}{4a\lambda}\cdot\frac{t^3}{3}$, ...(3)

$$y=\frac{k}{\lambda}\cdot\frac{t^2}{2}. \quad ...(4)$$

We have chosen that when $t=0$, both x and y vanish so that constants of integration in both the above integrations are zero.

Hence the equation of the path is obtained by eliminating t between (3) and (4).

$$x^2=\left(\frac{k^2}{12a\lambda}\right)^2 \cdot t^6=\left(\frac{k^2}{12a\lambda}\right)^2\cdot\left(\frac{2\lambda y}{k}\right)^3 \text{ by (3) and (4).}$$

Above is of the form $x^2=4by^3$, which we know represents a semi-cubical parabola.

Ex. 12. *A particle describes a parabola under gravity; show that the hodograph of its motion is a straight line parallel to the axis of the parabola and described with uniform velocity.*

The equations of motion of the particle describing a parabola under gravity are

$$\frac{d^2x}{dt^2}=0 \text{ and } \frac{d^2y}{dt^2}=-g. \quad ...(1)$$

Integrating, we get $\frac{dx}{dt}$=constant=c say. ...(2)

If (X, Y) be the co-ordinates of a point on the hodograph, then we know that $X=\lambda\frac{dx}{dt}$ and $Y=\lambda\frac{dy}{dt}$.

$$\therefore \quad X=\lambda c=a \text{ say by (2).}$$

Hence the locus of (X, Y) i.e. hodograph is the line $x=a$ which is parallel to the axis of y i.e. the axis of the

parabola described by a projectile which we know is parallel to the y-axis.

Again from § 1 we know that velocity at any point on the hodograph is equal to acceleration of the corresponding point on the curve *i. e.*

$$\frac{dX}{dt}=\lambda\frac{d^2x}{dt^2}=0 \quad \because \quad \frac{dx}{dt}=c \quad \text{by (1)}$$

and $$\frac{dY}{dt}=\lambda\frac{d^2y}{dt^2}=-\lambda g \quad \text{by (2).}$$

Hence the velocity in the hodograph is $-\lambda g$ which is constant which proves the 2nd part of the problem.

Ex. 13. *If a particle is projected initially with a velocity u and at an angle α to the horizontal, the resistance being k times the velocity, show that the hodograph of its path is the straight line $x(g+ku \sin \alpha)-(g+ky)\, u \cos \alpha=0$.*

Proceeding as in § 3 P. 271 (resisting medium) if (x, y) be the position of the particle at any time, then we have the following results no. (2) and (4) of § 3.

$$\frac{dx}{dt}=u\cos\alpha\,.\,e^{-kt} \text{ and } g+k\frac{dy}{dt}=(g+ku\sin\alpha)\,e^{-kt}.$$

Eliminating e^{-kt}, we get

$$\frac{x}{u\cos\alpha}=\frac{g+ky}{g+ku\sin\alpha}. \qquad ...(1)$$

Also we know that if (X, Y) be the co-ordinates of a point on the hodograph corresponding to a point (x, y) on the path, then

$$X=\lambda\frac{dx}{dt} \text{ and } Y=\lambda\frac{dy}{dt}. \qquad ...(2)$$

Hence from (1) by the help of (2), we get

$$\frac{X}{\lambda u\cos\alpha}=\frac{g+k\,.\,\frac{Y}{\lambda}}{g+ku\sin\alpha}.$$

$\therefore$ The equation of the hodograph is the locus of the point (X, Y) and is given by

$$x\,(g+ku\sin\alpha)-(\lambda g+ky)\,u\cos\alpha=0.$$

Ex. 14. *The resistance of air being supposed to vary as the cube of velocity, show that the hodograph of the motion of a projectile is $x^3+3xy^2=ay^3+b$, the axis of x being vertical.*

(Agra 64, 53)

Since x-axis is vertical and ψ is the angle which the tangent makes with x-axis *i. e.* vertical, proceeding as in § 4, P 280 or Ex. 6 P. 293, we write down the equations of motion along the horizontal and along the normal.

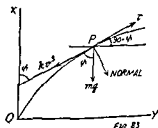


Fig 83

Let u stand for the horizontal component of the velocity at any point, so that from the figure it is clear that

$$u=v\sin\psi. \qquad \text{...(1)}$$

Also the equation of motion along the horizontal is

$$\frac{du}{dt}=-kv^3\sin\psi \qquad \text{...(2)}$$

[u being the horizontal velocity]

and along the normal the equation of motion is

$$\frac{v^2}{\rho}=g\sin\psi. \qquad \text{...(3)}$$

Here $\rho=\dfrac{ds}{d\psi}$ and not $-\dfrac{ds}{d\psi}$ as ψ increases with the increase of s.

From (1), $\quad \dfrac{du}{d\psi}\cdot\dfrac{d\psi}{dt}=-kv^3\sin\psi, \qquad \text{...(4)}$

From (3), $\quad v^2\dfrac{d\psi}{ds}=g\sin\psi$

$$v^2 \frac{d\psi}{dt} \cdot \frac{dt}{ds} = g \sin \psi$$

$$v \frac{d\psi}{dt} = g \sin \psi, \quad \because \quad \frac{ds}{dt} = v.$$

Putting for $\frac{d\psi}{dt}$ in (4), we get

$$\frac{du}{d\psi} \cdot \left(\frac{g \sin \psi}{v}\right) = -kv^3 \sin \psi.$$

$$\therefore \quad \frac{du}{d\psi} = -\frac{k}{g} v^4 = -\frac{k}{g} u^4 \operatorname{cosec}^4 \psi \text{ by (1).}$$

$$\therefore \quad \frac{du}{u^4} = -\frac{k}{g} \operatorname{cosec}^2 \psi \, (1 + \cot^2 \psi) \, d\psi.$$

Integrating, we get

$$-\frac{1}{3u^3} = \frac{k}{g} \left(\cot \psi + \frac{\cot^3 \psi}{3}\right) + A$$

or
$$\frac{1}{(v \sin \psi)^3} = -\frac{3k}{g} \left(\cot \psi + \frac{\cot^3 \psi}{3}\right) + B.$$

Above is the v, ψ relation of the motion of the particle nd its hodograph is obtained by replacing ψ by θ and $r = \lambda v$.

$$\therefore \quad \frac{\lambda^3}{r^3 \sin^3 \theta} = B - \frac{3k}{g} \left(\cot \theta + \frac{\cot^3 \theta}{3}\right).$$

Above gives us the polar equation of the hodograph and ve are to find the cartesian equation. Putting $x = r \cos \theta$, $y = r \sin \theta$ *i.e.* $\cot \theta = \frac{x}{y}$, we get the corresponding cartesian equation as

$$\frac{\lambda^3}{y^3} = B - \frac{3k}{g} \left(\frac{x}{y} + \frac{1}{3} \frac{x^3}{y^3}\right)$$

or
$$\frac{g}{k} \lambda^3 = B \cdot \frac{g}{k} y^3 - 3 \, (xy^2 + \tfrac{1}{3} x^3)$$

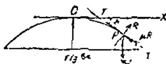
or
$$x^3 + 3xy^2 = B \cdot \frac{g}{k} y^3 - \frac{g}{k} \lambda^3.$$

Above is of the form $x^3 + 3xy^2 = ay^3 + b$. **Proved.**

Ex. 15. *A rough tube in the form of a cycloid is placed with its axis vertical and vertex upwards. A heavy particle is projected along the tube with such a velocity that its ratio to the velocity acquired in descending freely down the tube supposed smooth is as* $\sin\lambda : 1$, *where* $\tan\lambda$ *is the coefficient of friction. Prove that the hodograph is a circle.*

Refer § 4. (Constrained Motion) P. 221.

The equations of motion of the particle are



$$mv\frac{dv}{ds}=mg\sin\psi-\mu R \quad ...(1)$$

and $$m\frac{v^2}{\rho}=mg\cos\psi-R \quad ...(2)$$

Eliminating R between (1) and (2), we get

$$v\frac{dv}{ds}=g\sin\psi-\mu\left(g\cos\psi-\frac{v^2}{\rho}\right)$$

or $$\frac{1}{2}\frac{dv^2}{ds}-\mu\frac{v^2}{\rho}=g(\sin\psi-\mu\cos\psi).$$

Multiplying both sides by 2ρ and $\rho=\frac{ds}{d\psi}=4a\cos\psi$ for a cycloid whose intrinsic equation is $s=4a\sin\psi$,

$$\frac{dv^2}{ds}\cdot\frac{ds}{d\psi}-2\mu v^2=2g\cdot 4a\cos\psi(\sin\psi-\mu\cos\psi)$$

or $$\frac{dv^2}{d\psi}-2\mu v^2=8ag\cos\psi(\sin\psi-\mu\cos\psi). \quad ...(3)$$

Above is a linear differential equation and I.F.$=e^{-2\mu\psi}$.

Multiplying both sides by I. F. and integrating, we get

$$v^2 . e^{-2\mu\psi}=8ag\int e^{-2\mu\psi}\cos\psi(\sin\psi-\mu\cos\psi)\,d\psi+c. \quad ...(4)$$

Put $e^{-\mu\psi}.(\sin\psi-\mu\cos\psi)=z$. (See P. 222).

$\therefore$ $[-\mu e^{-\mu\psi}(\sin\psi-\mu\cos\psi)+e^{-\mu\psi}(\cos\psi+\mu\sin\psi)]\,d\psi=dz$

or $$(1+\mu^2)\,e^{-\mu\psi}\cos\psi=dz.$$

Now (4) can be written as

$$v^2e^{-2\mu\psi}=8ag\int e^{-\mu\psi}(\sin\psi-\mu\cos\psi)\,.\,e^{-\mu\psi}\cos\psi\,d\psi+C$$

or $$v^2e^{-2\mu\psi}=8ag\int z\frac{dz}{1+\mu^2}+C$$

or $$v^2e^{-2\mu\psi}=\frac{8ag}{1+\mu^2}\frac{z^2}{2}+C$$

or $$v^2e^{-2\mu\psi}=\frac{4ag}{1+\mu^2}\,e^{-2\mu\psi}(\sin\psi-\mu\cos\psi)^2+C. \quad ...(6)$$

Initially when $\psi=0$, then $v=\sin\lambda$ (velocity acquired in falling down the tube to cusp a vertical distance equal to $2a$ when the tube is smooth. By energy equation this velocity is $\sqrt{(2g\,.\,2a)}$ or $2\sqrt{(ga)}$.

$\therefore$ Initially when $\psi=0$, $v=\sin\lambda\,.\,2\sqrt{(ga)}$. ...(7)

Hence from (6) and (7), we get

$$\sin^2\lambda\,.\,4ga\,.\,1=\frac{4ag}{1+\tan^2\lambda}\,.\,1\,(0-\mu)^2+C$$

or $$\sin^2\lambda=\cos^2\lambda\,.\frac{\sin^2\lambda}{\cos^2\lambda}+C \quad \text{or} \quad C=0,\ \because\ \mu=\tan\lambda.$$

Hence from (6) on putting $C=0$, we get

$$v^2=\frac{4ag}{1+\mu^2}(\sin\psi-\mu\cos\psi)^2$$

or $$v=2\sqrt{(ag)}\left(\sin\psi\,.\frac{1}{\sqrt{(1+\mu^2)}}-\frac{\mu}{\sqrt{(1+\mu^2)}}\cos\psi\right)$$

or $$v=2\sqrt{(ag)}\,(\sin\psi\cos\lambda-\cos\psi\sin\lambda)$$

or $$v=2\sqrt{(ag)}\sin(\psi-\lambda).$$

Above is the v, ψ relation of the motion of the particle and its hodograph is obtained by putting θ for ψ and $r=\lambda v$.

$$\therefore\ \text{hodograph is } \frac{r}{\lambda}=2\sqrt{(ag)}\sin(\theta-\lambda)$$

or $$r=2\lambda\sqrt{(ag)}\sin(\theta-\lambda)$$

or $$r=d\sin(\theta-\lambda) \quad \text{which represents a circle.}$$

Ex. 16. *If a circle be described under a constant acceleration not tending to the centre, prove that the hodograph is a lemniscate.* (Agra 48)

We are given that the circle is described with a constant acceleration say d. Also in a circle $\rho=\frac{ds}{d\psi}=a$.

$$\therefore \quad \left[\left(v\frac{dv}{ds}\right)^2+\left(\frac{v^2}{\rho}\right)^2\right]=d$$

or
$$\left(v\frac{dv}{d\psi}\cdot\frac{d\psi}{ds}\right)^2+\left(\frac{v^2}{a}\right)^2=d^2$$

or
$$\left(v\frac{dv}{d\psi}\right)^2+v^4=a^2d^2=k^4 \text{ say.}$$

$$\therefore \quad v\frac{dv}{d\psi}=\pm\sqrt{(k^4-v^4)}.$$

$$\therefore \quad \frac{2v\,dv}{\sqrt{(k^4-v^4)}}\,dv=2d\psi, \text{ taking + ve sign.}$$

$$\therefore \quad \sin^{-1}\frac{v^2}{k^2}=2\psi+B.$$

Let us suppose that when $\psi=0$, $v^2=k^2$.

$$\therefore \quad B=\sin^{-1}1=\pi/2.$$

$$\therefore \quad \frac{v^2}{k^2}=\sin\left(\frac{\pi}{2}+2\psi\right)=\cos 2\psi.$$

Above is the v, ψ relation of the motion of the particle and in order to find its hodograph replace ψ by θ and $r=\lambda v$.

$$\therefore \quad \frac{r^2}{\lambda^2k^2}=\cos 2\theta \quad \text{or} \quad r^2=\lambda^2k^2\cos 2\theta$$

or $\quad r^2=b^2\cos 2\theta$, which represents a lemniscate.

Had we taken the negative sign with the radical we would have obtained $\cos^{-1}\frac{v^2}{k^2}=2\psi+B$.

Then we would have chosen that as $\psi=0$, $v^2=k^2$.

$$\therefore \quad 2B=0\;; \quad \therefore \quad \frac{v^2}{k^2}=\cos 2\psi.$$

and we would have obtained the same answer.

Ex. 17 *A particle moves under a central acceleration μu^{2n+3} being projected from an apse at a distance a with a velocity $\sqrt{\left(\frac{\mu}{n+1}\right)}\cdot\frac{1}{a^{n+1}}$. Show that the hodograph is $r^m \cos m\theta = const.$, where $m=\frac{n}{n+1}$.*

Proceeding as in Ex. 1 (P. 24 Chapter of Central Forces) the equation of the path is $r^n = a^n \cos n\theta$. Taking log,

$$n \log r = n \log a + \log \cos n\theta; \quad \therefore \quad \frac{n}{r}\frac{dr}{d\theta} = -n \tan n\theta.$$

$$\therefore \quad \cot \phi = -\tan n\theta = \cot\left(\frac{\pi}{2} + n\theta\right); \quad \therefore \quad \phi = \frac{\pi}{2} + n\theta,$$

$$\psi = \theta + \phi = (n+1)\theta + \frac{\pi}{2} \text{ or } \theta = \frac{\psi - \pi/2}{n+1}.$$

$$\therefore \quad n\theta = \frac{n\left(\psi - \frac{\pi}{2}\right)}{n+1} = m\psi - m\frac{\pi}{2}; \quad \because \quad m = \frac{n}{n+1},$$

$$p = r \sin \phi \quad \text{or} \quad p = a(\cos n\theta)^{1/n} \cos n\theta$$

or $$p = a \cos (n\theta)^{\frac{n+1}{n}} = a \cos^{1/m}\left(m\psi - m\frac{\pi}{2}\right).$$

But in a central orbit, $vp = h$.

$$\therefore \quad \frac{h}{v} = a \cos^{1/m}\left(m\psi - m\frac{\pi}{2}\right).$$

Above is v, ψ relation. Write θ for ψ and $r = \lambda v$.

$$\therefore \quad \frac{h}{a} = \frac{r}{\lambda} \cos^{1/m}\left(m\theta - \frac{m\pi}{2}\right).$$

Raise both sides to the power m.

$$\therefore \quad \text{constant} = r^m \cos\left(m\theta - \frac{m\pi}{2}\right).$$

Now turning through a right angle, we get the hodograph as $$r^m \cos m\theta = \text{constant}.$$



CHAPTER VI

REVOLVING CURVES

§ 1. Motion on a revolving curve.

A given curve turns in its own plane about a given fixed point O with angular velocity ω; a small bead P moves on the curve under the action of given forces whose components along and perpendicular to OP are X and Y; to discuss the motion.

(Punjab 56, 57, 60; Rajputana 65, 63; Agra 57, 59)

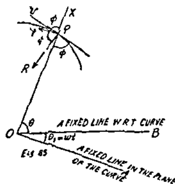


Fig 85

Let OA be a fixed line in the plane of the curve and OB be a fixed line with respect to the curve, *i. e.* OB is fixed in the curve. In other words, it means that as the curve revolves about O with angular velocity ω, then OB also revolves with the same angular velocity. Let us suppose that initially OB coincides with OA so that after time t, $\angle AOB=\theta_1$ $=\omega t$, where θ_1 is the angle through which the curve revolves about the fixed line OA w[illegible] velo[illegible]

Again let θ be the [illegible] the [illegible] OP to the curve makes with O[illegible] the curve so that $\angle BOP=\theta$ [illegible] OB a line fixed w. r. t. [illegible] co-ordinates w. r. t. OA are $(r, \theta$[illegible] θ_1—

X and Y are the components of forces along and perpendicular to the radius vector OP. Let R be the normal reaction of the bead.

We know that radial and cross-radial accelerations of a particle are $\ddot{r}-r\dot{\theta}^2$ and $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$. Here θ is $\theta+\theta_1$ where $\theta_1=\omega t$.

The radial and cross-radial equations of motion of the bead are

$$m\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}+\frac{d\theta_1}{dt}\right)^2\right]=X-R\sin\phi, \quad \because \quad \theta=\theta+\theta_1.$$

But $\theta_1=\omega t$; $\therefore \quad \frac{d\theta_1}{dt}=\omega$.

$$\therefore \quad \frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}+\omega\right)^2=\frac{X}{m}-\frac{R}{m}\sin\phi \qquad \text{...(1)}$$

and
$$m\left[\frac{1}{r}\frac{d}{dt}\left\{r^2\frac{d}{dt}(\theta+\theta_1)\right\}\right]=Y+R\cos\phi$$

or
$$\frac{1}{r}\frac{d}{dt}\left\{r^2\frac{d\theta}{dt}+r^2\omega\right\}=\frac{Y}{m}+\frac{R}{m}\cos\phi. \qquad \text{...(2)}$$

Equations (1) and (2), after simplification can be easily put as

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=\frac{X}{m}-\frac{R}{m}\sin\phi+r\omega^2+2r\omega.\frac{d\theta}{dt} \qquad \text{...(3)}$$

and
$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=\frac{Y}{m}+\frac{R}{m}\cos\phi-\frac{1}{r}\frac{d}{dt}(r^2\omega)$$

or
$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=\frac{Y}{m}+\frac{R}{m}\cos\phi-2\omega.\frac{dr}{dt}. \qquad \text{...(4)}$$

Now suppose that v is the velocity of the bead w. r. t. the curve *i.e.* w. r. t. OB fixed w. r. t. curve ; then

$$\frac{dr}{dt}=\text{radial velocity}=v\cos\phi,$$

$$r\frac{d\theta}{dt}=\text{transverse velocity}=v\sin\phi.$$

Using the above relations, equations (3) and (4) change to

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=\frac{X}{m}-\frac{R}{m}\sin\phi+r\omega^2+2\omega.v\sin\phi$$

$$=\frac{X}{m}-\frac{1}{m}(R-2m\omega v)\sin\phi+\frac{mr\omega^2}{m}$$

$$=\frac{X}{m}-\frac{R'}{m}\sin\phi+\frac{mr\omega^2}{m}$$

where $\qquad R'=R-2m\omega$

or $$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=\frac{X+mr\omega^2}{m}-\frac{R'}{m}\sin\phi$$

$$=\frac{X'}{m}-\frac{R'}{m}\sin\phi, \qquad \ldots(5)$$

where $\mathbf{X'=X+mr\omega^2}$ and $\mathbf{R'=R-2m\omega v}$ or $\mathbf{R=R'+2m\omega v}$.

Similarly equation (4) becomes

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=\frac{Y}{m}+\frac{R}{m}\cos\phi-2\omega v\cos\phi$$

$$=\frac{Y}{m}+\frac{R-2mv\omega}{m}\cos\phi$$

$$=\frac{Y}{m}+\frac{R'}{m}\cos\phi \qquad \ldots(6)$$

where $\qquad \mathbf{R'=R-2mv\omega} \quad$ or $\quad \mathbf{R=R'+2mv\omega}$.

Now suppose that the curve is fixed *i e.* $\theta_1=0$; then the corresponding equations of motion under the same forces X and Y are obtained by putting $\theta_1=0$ in (1) and (2) and they are

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=\frac{X}{m}-\frac{R}{m}\sin\phi \qquad \ldots(7)$$

and $$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=\frac{Y}{m}+\frac{R}{m}\cos\phi. \qquad \ldots(8)$$

If we compare (5) with (7) and (6) with (8), we can say

that the equations of motion of the bead when the curve is revolving with angular velocity ω are of same form as when the curve is treated as fixed. But

$$X' = X + mr\omega^2$$

and
$$R' = R - 2m\omega v$$

Above shows that if we apply an additional force $mr\omega^2$ along the radius vector OP away from pole and take the reaction of the bead to be R', then we can write down the equations of motion taking the curve to be stationary. One thing should be clearly observed that in order to get the reaction R (when the curve is revolving), we will have to add $2m\omega v$ to the value of R' in order to get R.

$$\therefore \quad R' = R - 2m\omega v \quad \text{or} \quad R = R' + 2m\omega v.$$

The above process is called reducing the curve to rest :

Rule. Introduce along the radius vector a force equal to $mr\omega^2$ in addition to the given forces and take the reaction to be R' and now treat the question as if the curve were fixed. It is not necessary to write the radial and transverse equations of motion once the curve is reduced to rest. You may resolve the forces in any manner you like *i.e.* you may even write tangential and normal equations of motion. Another point to be noted is that the value of R' will not be true normal reaction. The true value of $R = R' + 2m\omega v$, where v is the velocity of the bead **w. r. t. the tube (Note)** and will be +ive if its direction is the same as that in which the tube revolves. If however the direction of the bead and that of the tube be opposite, *i. e.* one moves clockwise and the other anti-clockwise, then v will be $-$ive and in that case true normal reaction $R = R' + 2m\omega(-v)$.

Note. If v be the velocity and $\omega = \frac{d\theta}{dt}$ be the angular

velocity, then $\qquad \dfrac{d\theta}{dt}=\dfrac{v \sin\phi}{r}=\dfrac{v.p}{r^2} \quad \because \; p=r \sin\phi$

or we may say $r\dfrac{d\theta}{dt}=v \sin \phi=$ transverse velocity.

Exercise

Ex. 1. *A smooth plane tube revolving with angular velocity* ω *about a point O in its plane contains a particle of mass m, which is acted upon by a force* $mr\omega^2$ *towards O. Show that the reaction of the tube is* $A+\dfrac{B}{\rho}$ *where A and B are constants and* ρ *is the radius of curvature of the tube at the point occupied by the particle.*

(Indore 66; Rajputana 62 ; Agra 50, 57; Punjab 56

Let R be the normal reaction when the tube is revolving and the corresponding normal reaction when the curve is reduced to rest be R' so that $R=R'+2m\omega v$. The curve is reduced to rest by introducing a force $mr\omega^2$ along OP. Also we are given that the particle is acted upon by a force $mr\omega^2$ along PO. So the net force acting on the particle is R' only along the normal at P, the curve being taken to be stationary.

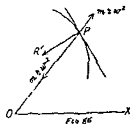


Fig 86

Writing the tangential and normal equations of motion,

we have $$mv\frac{dv}{ds}=0 \quad \text{or} \quad v\frac{dv}{ds}=0. \qquad ...(1)$$

There being no force along the tangent as R' is along the normal and the other two forces $mr\omega^2$ being equal and opposite cancel each other.

Also $$m\,\frac{v^2}{\rho}=R'. \qquad ...(2)$$

From (1), we get on integrating that $v^2=\text{constant}=k^2$ say.

$\therefore$ From (2), $\qquad R'=\dfrac{m}{\rho}.k^2.$

$\therefore$ Actual reaction R when the curve is revolving

$$=R'+2m\omega v.$$

$$\therefore \quad R=\frac{m}{\rho}k^2+2m\omega v=\frac{m}{\rho}k^2+2m\omega k, \quad \because \quad v^2=k^2$$

or $\quad R=\dfrac{B}{\rho}+A \quad$ where A and B are constants.

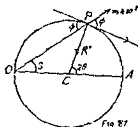
Ex. 2. *A smooth circular tube contains a particle of mass m and lies on a smooth table. The tube starts rotating with constant angular velocity ω about an axis perpendicular to the plane of the tube which passes through the other end, O, of the diameter through the intial position, A, of the particle. Show that in time t the particle will have described an angle φ about the centre of tube equal to 4 $\tan^{-1}\tanh(\frac{1}{2}\omega t)$. Show also that the reaction between the tube and the particle is then equal to*

$$2ma\omega^2\cos\frac{\phi}{2}\left(3\cos\frac{\phi}{2}-2\right).$$

(Agra 52, 55, 58, 65)

Equation of the circle referred to O as pole and OA as diameter is

$$r=2a\cos\theta. \qquad ...(1)$$



Introduce a force $mr\omega^2$ along the radius vector OP and the normal reaction be R', where $R'=R-2mv\omega$, R being the true reaction when the curve is revolving and v the velocity of the particle relative to tube.

Equations of motion along the tangent and normal are

$$mv\frac{dv}{ds}=mr\omega^2.\cos\phi=mr\omega^2.\frac{dr}{ds}$$

or $\quad v\,dv=\omega^2 r\,dr \quad$ or $\quad v^2=\omega^2r^2+A. \qquad ...(2)$

Initially $r=2a$ at A as the distance of A from O is $2a$. The particle will describe a circle of radius $2a$ with angular

velocity ω about O. Hence its velocity along the tangent at A will be $2a\omega$.

$\because$ in a circle $s=a\theta$, $\therefore$ $v=\frac{ds}{dt}=\frac{ds}{d\theta}\cdot\frac{d\theta}{dt}=a\frac{d\theta}{dt}=\text{radius}.\omega$

Putting $r=2a$ and $v=2a\omega$ in (2), we get

$$(2a\omega)^2=\omega^2.4a^2+A; \quad \therefore \quad A=0.$$

$$\therefore \quad v^2=\omega^2 r^2; \quad \therefore \quad v=\omega r$$

or $$\frac{ds}{dt}=\omega.2a\cos\theta \text{ by (1) or } \frac{ds}{d\theta}\cdot\frac{d\theta}{dt}=\omega.2a\cos\theta.$$

But $$\frac{ds}{d\theta}=\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}=\sqrt{(4a^2\cos^2\theta+4a^2\sin^2\theta)}=2a.$$

$$\therefore \quad 2a\frac{d\theta}{dt}=\omega.2a\cos\theta; \quad \therefore \quad \omega\,dt=\sec\theta\,d\theta.$$

Integrating, $\omega t=\log(\sec\theta+\tan\theta)+B$.

Initially when $t=0$, $\theta=0$; $\therefore$ $B=\log 1=0$.

$$\therefore \quad \sec\theta+\tan\theta=e^{\omega t}. \quad ...(3)$$

$$\therefore \quad \frac{1}{\sec\theta+\tan\theta}=e^{-\omega t} \text{ or } \frac{\sec\theta-\tan\theta}{\sec^2\theta-\tan^2\theta}=e^{-\omega t}$$

or $$\sec\theta-\tan\theta=e^{-\omega t}. \quad ...(4)$$

Adding (3) and (4), we get

$$2\sec\theta=e^{\omega t}+e^{-\omega t}=2\cosh\omega t.$$

$$\therefore \quad \cosh\omega t=\sec\theta=\frac{1}{\cos\theta}$$

or $$\frac{\cosh^2\frac{\omega t}{2}+\sinh^2\frac{\omega t}{2}}{\cosh^2\frac{\omega t}{2}-\sinh^2\frac{\omega t}{2}}=\frac{1}{\cos\theta}.$$

Applying componendo and dividendo,

$$\frac{2\sinh^2\frac{\omega t}{2}}{2\cosh^2\frac{\omega t}{2}}=\frac{1-\cos\theta}{1+\cos\theta}=\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}.$$

$$\therefore \quad \tan\frac{\theta}{2}=\tanh\frac{\omega t}{2} \quad \text{or} \quad \theta=2\tan^{-1}\left(\tanh\frac{\omega t}{2}\right). \quad ...(5)$$

Since the angle at the centre is twice the angle subtended at a point O on the circumference, hence the angle ϕ described about the centre is 2θ.

$$\therefore \quad \phi=2\theta=4\tan^{-1}\left(\tanh\frac{\omega t}{2}\right).$$

Again normal equation of motion is

$$m\frac{v^2}{\rho}=R'-mr\omega^2\sin\phi.$$

Put $\sin\phi=r\dfrac{d\theta}{ds}$ and $\dfrac{ds}{d\theta}=2a$ and $\rho=a$ and $v^2=\omega^2r^2$.

$$\therefore \quad R'=m\frac{\omega^2r^2}{a}+\frac{mr^2\omega^2}{2a}=\frac{3mr^2\omega^2}{2a}. \qquad \text{...(6)}$$

$\therefore$ actual reaction R when the curve is revolving is given by $\quad R'=R-2m\omega v \quad$ or $\quad R=R'+2m\omega v. \qquad$ ...(7)

In the above formula, v stands for the velocity of the bead with respect to the curve and it is $+$ive if their directions are the same. But in our problem the directions of motion of the bead and the curve are opposite and as such v will be taken as $-$ive in result (7). (See Rule § 1)

Because if the particle were actually at rest when the motion began then its velocity v with respect to the curve must be in the opposite direction.

$$\therefore \quad R=R'+2m\omega(-v)$$

or
$$R=\frac{3mr^2\omega^2}{2a}+2m\omega(-\omega r). \quad \because \; v^2=\omega^2r^2$$

$$=mr\omega^2\left(\frac{3r}{2a}-2\right)$$

$$=m\omega^2.2a\cos\theta\,(3\cos\theta-2), \quad \because \; r=2a\cos\theta.$$

Since we are to put the answer in terms of ϕ, where $\phi=2\theta$,

$$\therefore \quad R=2am\omega^2\cos\frac{\phi}{2}\left(3\cos\frac{\phi}{2}-2\right). \qquad \textbf{Proved.}$$

Ex. 3. *A particle moves in a wire in the form of a circle of radius a constrained to rotate in its own plane with angular velocity ω about a point O of itself. The particle is initially at the end of the diameter through O and is projected with velocity 2bω relative to the wire Show that the particle describes a quadrant of the circle in time*

$$\frac{\pi a}{4b\omega}\left[1+\frac{a^2}{4b^2}\left(1-\frac{2}{\pi}\right)\right] \text{ where } b \text{ is large.} \quad \text{(Agra 64)}$$

Proceeding exactly as is Ex. 2 and with the same figure we get that $v\,dv=\omega^2 r\,dr$, $\therefore\ v^2=\omega^2 r^2+A$.

Initially when the particle is at A, the other end of diameter through O, its velocity v relative to the wire is $2b\omega$ given. Also at A, $r=2a$.

$$\therefore\ 4b^2\omega^2=\omega^2.4a^2+A\ ;\ \therefore\ A=4\omega^2(b^2-a^2).$$

$$\therefore\ v^2=\omega^2r^2+4\omega^2(b^2-a^2). \quad \text{Put } r=2a\cos\theta.$$

$$\therefore\ v^2=4\omega^2[b^2-a^2(1-\cos^2\theta)]=4\omega^2(b^2-a^2\sin^2\theta).$$

$$\therefore\ v=\frac{ds}{dt}=2\omega.\sqrt{(b^2-a^2\sin^2\theta)}$$

or $\quad \dfrac{ds}{d\theta}\cdot\dfrac{d\theta}{dt}=2\omega\sqrt{(b^2-a^2\sin^2\theta)}$. Put $\dfrac{ds}{d\theta}=2a$ as in Ex. 2.

$$\therefore\ 2a\frac{d\theta}{dt}=2\omega\sqrt{(b^2-a^2\sin^2\theta)}.$$

$$\therefore\ \frac{a}{\omega}\frac{d\theta}{\sqrt{(b^2-a^2\sin^2\theta)}}=dt. \qquad ..(1)$$

When $t=0$, the particle is at A where $\theta=0$; when $t=T$ the particle is at the end of a quadrant so that the angle at the centre is $\frac{\pi}{2}$, *i. e.* angle θ at the pole O is $\frac{\pi}{4}$.

Integrating (1) between the above limits, we get

$$\int_0^{\pi/4}\frac{a}{\omega b}\frac{d\theta}{\left(1-\frac{a^2}{b^2}\sin^2\theta\right)^{1/2}}=\int_0^T dt.$$

or $$\frac{a}{\omega b}\int_0^{\pi/4}\left(1-\frac{a^2}{b^2}\sin^2\theta\right)^{-1/2}d\theta=\Big[\ t\ \Big]_0^T$$

or $$\frac{a}{\omega b}\int_0^{\pi/4}\left(1+\frac{1}{2}\frac{a^2}{b^2}\sin^2\theta\ldots\right)d\theta=T.$$

We have neglected higher terms as b is large, *i.e.* $\frac{a}{b}$ is small.

$$\therefore\ \frac{a}{\omega b}\int_0^{\pi/4}\left[1+\frac{1}{2}\frac{a^2}{b^2}\left(\frac{1-\cos 2\theta}{2}\right)\right]d\theta=T$$

or $$\frac{a}{\omega b}\left[\left(1+\frac{1}{4}\frac{a^2}{b^2}\right)\theta-\frac{1}{4}\frac{a^2}{b^2}\frac{\sin 2\theta}{2}\right]_0^{\pi/4}=T\ .$$

or $$\frac{a}{\omega b}\left[\left(1+\frac{1}{4}\frac{a^2}{b^2}\right)\frac{\pi}{4}-\frac{1}{4}\frac{a^2}{b^2}\cdot\frac{1}{2}\right]=T$$

or $$\frac{\pi a}{4\omega b}\left[1+\frac{a^2}{4b^2}\left(1-\frac{2}{\pi}\right)\right]=T.$$ Proved.

Ex. 4. *A bead is at rest at the vertex of a smooth wire in the shape of a cardioide $r=a\,(1+\cos\theta)$ The wire begins to move about its pole with uniform angular velocity ω Show that the pressure will vanish when the bead has described an angle $2\cos^{-1}(\frac{4}{5})$ about the pole. Show also that the bead moves towards the pole but never reaches it and that the distance of the bead from the pole at time t is $r=2a\,\text{sech}^2\left(\frac{\omega t}{2}\right)$.*

(Agra 48)

Certain results of differential calculus connected with the equation $r=a\,(1+\cos\theta)$.

Taking log, $\log r=\log a+\log(1+\cos\theta)$.

$$\therefore\ \frac{1}{r}\frac{dr}{d\theta}=-\frac{\sin\theta}{1+\cos\theta}\quad\text{or}\quad\cot\phi=-\tan\frac{\theta}{2}.$$

$$\therefore\ \phi=\frac{\pi}{2}+\frac{\theta}{2},\ \ldots(A)\ p=r\sin\phi=r\cos\frac{\theta}{2}.$$

But $\qquad r=a(1+\cos\theta)=2a\cos^2\frac{\theta}{2}.$

$\therefore \quad p=r.\sqrt{\left(\frac{r}{2a}\right)}$ or $2ap^2=r^3$ is the pedal equation.

Again $\qquad \frac{dp}{dr}=\frac{3}{2}\sqrt{\left(\frac{r}{2a}\right)}.$

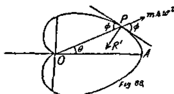
$$\therefore \quad \rho=r\frac{dr}{dp}=r.\frac{2}{3}\sqrt{\left(\frac{2a}{r}\right)}=\frac{2}{3}\sqrt{(2ar)}. \qquad ...(B)$$

Also $\qquad \sin\phi=r\frac{d\theta}{ds}$ or $\cos\frac{\theta}{2}=2a\cos^2\frac{\theta}{2}\frac{d\theta}{ds}.$

$$\therefore \quad \frac{ds}{d\theta}=2a\cos\frac{\theta}{2}. \qquad ...(C)$$

Introduce a force $mr\omega^2$ along the radius vector OP in order to reduce the curve to rest. If R' be the reaction, then

$$R'=R-2mv\omega, \qquad ...(1)$$



R being the true reaction when the curve is revolving and v is the velocity of the particle relative to the curve which is +ive if the directions of the motion of the particle and the curve are same. In case they are opposite, then v is $-$ive. Since the bead is actually at rest at A, when motion ensued, therefore its velocity w. r. t. the curve should be in opposite direction. Here we shall take v to be $-$ive in (1), *i.e.* $R'=R-2m\omega(-v)$. ...(2)

Now writing down the tangential and normal equations of motion, we have the following :

$$mv\frac{dv}{ds}=mr\omega^2\cos\phi=mr\omega^2.\frac{dr}{ds}.$$

$$\therefore \quad v\,dv=r\omega^2\,dr.$$

Integrating, we get $\quad v^2=r^2\omega^2+A.$

Initially when at A where $\theta=0$ *i.e.* $r=2a$, and also $\phi=\frac{\pi}{2}+\frac{\theta}{2}=\frac{\pi}{2}$, then angular velocity$=\frac{v \sin \phi}{r}$

or $$\omega=\frac{v}{2a}\cdot 1. \quad \therefore \quad v=2a\omega.$$

$$\therefore \quad (2a\omega)^2=4a^2\omega^2+A. \quad \therefore \quad A=0.$$

Hence $$v^2=r^2\omega^2 \quad \text{or} \quad v=r\omega \qquad ...(3)$$

or $$\frac{ds}{dt}=r\omega \quad \text{or} \quad \frac{ds}{d\theta}\cdot\frac{d\theta}{dt}=r\omega.$$

But $\frac{ds}{d\theta}=2a\cos\frac{\theta}{2}$ and $r=a\,(1+\cos\theta)=2a.\cos^2\frac{\theta}{2}$.

$$\therefore \quad 2a\cos\frac{\theta}{2}\,\frac{d\theta}{dt}=2a\cos^2\frac{\theta}{2}.\omega$$

or $$\omega\,dt=\int \sec\frac{\theta}{2}\,d\theta.$$

Integrating, we get

$$\omega t=2\log\left(\sec\frac{\theta}{2}+\tan\frac{\theta}{2}\right). \qquad ...(4)$$

The constants of integration vanish because when $t=0$, $\theta=0$.

$$\therefore \quad e^{\omega t/2}=\sec\frac{\theta}{2}+\tan\frac{\theta}{2}.$$

$$\therefore \quad e^{-\omega t/2}=\frac{1}{\sec\frac{\theta}{2}+\tan\frac{\theta}{2}}=\frac{\sec\frac{\theta}{2}-\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}-\tan^2\frac{\theta}{2}}=\sec\frac{\theta}{2}-\tan\frac{\theta}{2}.$$

$$\therefore \quad \frac{e^{\omega t/2}+e^{-\omega t/2}}{2}=\sec\frac{\theta}{2} \quad \text{or} \quad \cosh\frac{\omega t}{2}=\sec\frac{\theta}{2}.$$

or $$\cos\frac{\theta}{2}=\text{sech}\,\frac{\omega t}{2} \quad \therefore \quad 2a\cos^2\frac{\theta}{2}=2a\,\text{sech}^2\,\frac{\omega t}{2}$$

or $$a\,(1+\cos\theta)=2a\,\text{sech}^2\,\frac{\omega t}{2}$$

or $$r=2a\ \text{sech}^2\frac{\omega t}{2}.$$

Result (5) gives the distance of the bead from the pole at time t. Again the bead will reach the pole when $\theta=\pi$ because at pole $r=0$, *i.e.* $a(1+\cos\theta)=0$, *i.e.* $\cos\theta=-1$ or $\theta=\pi$.

Putting $\theta=\pi$ in (4), we get

$$\omega t=2\log(\infty)=\infty. \quad \therefore \quad t=\text{infinite}.$$

Above result shows that the bead will never reach the pole.

Again normal equation of motion is

$$m\frac{v^2}{\rho}=R'-mr\omega^2\sin\phi.$$

Put $$\sin\phi=\frac{p}{r},\ v^2=\omega^2r^2,\ \rho=\frac{2}{3}\sqrt{(2ar)}.$$

$$\therefore\quad R'=m\frac{\omega^2r^2}{\frac{2}{3}\sqrt{(2ar)}}+mr\omega^2.\frac{p}{r}.\quad \text{But}\quad p=r\sqrt{\left(\frac{r}{2a}\right)}.$$

$$\therefore\quad R'=mr\omega^2\left[\frac{3}{2}\sqrt{\left(\frac{r}{2a}\right)}+\sqrt{\left(\frac{r}{2a}\right)}\right]$$

$$=mr\omega^2.\frac{5}{2}\sqrt{\left(\frac{r}{2a}\right)}.$$

From (2), $R'=R-2m\omega(-v)=R+2m\omega(v)$.

$$\therefore\quad R=R'-2m\omega(v)$$

$$=mr\omega^2.\frac{5}{2}\sqrt{\left(\frac{r}{2a}\right)}-2m\omega(r\omega)\ \text{from (3) } i.e.\ v=r\omega$$

or $$R=mr\omega^2\left[\frac{5}{2}\sqrt{\left(\frac{r}{2a}\right)}-2\right].$$

Now reaction R will vanish, when $\frac{5}{2}\sqrt{\left(\frac{r}{2a}\right)}=2$

or $$25r=32a \quad \text{or} \quad 25a(1+\cos\theta)=32a$$

or $\quad 25 \cdot 2\cos^2\frac{\theta}{2}=32$; $\therefore\ \cos^2\frac{\theta}{2}=\frac{16}{25}$

or $\quad \cos\frac{\theta}{2}=\frac{4}{5}$ or $\theta=2\cos^{-1}\frac{4}{5}$. **Proved.**

Ex. 5. *A particle is free to move along a smooth curve $r^n=a^n\cos n\theta$ which revolves in its own plane round the origin with uniform angular velocity ω. Prove that if the particle be initially at rest in the position $\theta=0$ and there are no external forces, its distance at time t is $a\,\{\text{sech}\,(n\omega t)\}^{1/n}$.* **(Rajputana 63)**

Also prove that the pressure will vanish when the distance is $a\left\{\frac{2}{n+2}\right\}^{1/n}$ and the time taken then is

$$\frac{2}{n\omega}\sinh^{-1}(\tfrac{1}{2}\sqrt{n}).$$

Exactly same type as Q. 4.

Certain important results of differantial calculus connected with the equation $r^n=a^n\cos n\theta$.

Taking log, we get $n\log r=n\log a+\log\cos n\theta$.

Differentiating, we get $n.\frac{1}{r}\frac{dr}{d\theta}=-n\frac{\sin n\theta}{\cos n\theta}$

or $\quad \cot\phi=-\tan n\theta=\cot\left(\frac{\pi}{2}+n\theta\right)$; $\therefore\ \phi=\frac{\pi}{2}+n\theta$...(*A*)

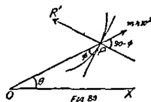
$$p=r\sin\phi=r\cos n\theta=r.\frac{r^n}{a^n}=\frac{r^{n+1}}{a^n}.$$

$$\therefore\ \frac{dp}{dr}=(n+1)\frac{r^n}{a^n}.$$

$$\therefore\ \rho=r\frac{dr}{dp}=r.\frac{a^n}{(n+1)\,r^n}=\frac{a^n}{(n+1)\,r^{n-1}}. \quad ...(B)$$

Also $\sin\phi=r\frac{d\theta}{ds}$ or $\cos n\theta=r\frac{d\theta}{ds}$.

$$\therefore\ \frac{ds}{d\theta}=r\sec n\theta. \quad ...(C)$$



Introduce a force $mr\omega^2$ along the radius vector OP in order to reduce the curve to rest. If R' be the reaction, then

$$R'=R-2mv\omega, \qquad ...(1)$$

R being the true reaction when the curve is revolving and v the velocity of the particle relative to the curve which is +ive if the direction of motion of the particle and the curve are the same. In case they are opposite, then v is −ive. Since the particle is actually at rest at $\theta=0$ when motion ensued, therefore its velocity w. r. t. the curve should be in opposite direction.

Hence we shall take v to be −ive in (1)

i.e. $$R'=R-2m\omega\,(-v). \qquad ...(2)$$

Writing down the tangential equation of motion, we get

$$mv\frac{dv}{ds}=mr\omega^2 \cos\phi=mr\omega^2\frac{dr}{ds}.$$

$$\therefore\quad v\,dv=r\omega^2\,dr.$$

Integrating, we get $v^2=r^2\omega^2+A$.

Initially when $\theta=0$ *i. e.* $r=a$, $\phi=\frac{\pi}{2}+n\theta=\frac{\pi}{2}$.

Then angular velocity$=\frac{v\sin\phi}{r}$ or $\omega=\frac{v.1}{a}$; $\therefore$ $v=a\omega$.

$$\therefore\quad (a\omega)^2=a^2\omega^2+A\ ;\quad \therefore\quad A=0.$$

Hence $$v^2=r^2\omega^2 \text{ or } v=r\omega \qquad ...(3)$$

or $$\frac{ds}{dt}=r\omega \text{ or } \frac{ds}{d\theta}\cdot\frac{d\theta}{dt}=\omega r$$

or $$(r\sec n\theta)\cdot\frac{d\theta}{dt}=\omega r \text{ by (C) or } \sec n\theta\frac{d\theta}{dt}=\omega.$$

Integrating, $\int n\omega\,dt=\int n\sec n\theta\,d\theta.$

Integrating, $n\omega t = \log(\sec n\theta + \tan n\theta) + B$.

When $t=0$, $\theta=0$; $\therefore$ $B=0$.

$$\therefore \quad \sec n\theta + \tan n\theta = e^{n\omega t},$$

$$\sec n\theta - \tan\theta = e^{-n\omega t} \text{ as in Q. 3 and 4.}$$

Adding, we get $\sec n\theta = \dfrac{e^{n\omega t} + e^{-n\omega t}}{2} = \cosh n\omega t$

or $\quad \cos n\theta = \text{sech } n\omega t$

or $\quad r^n = a^n \text{ sech } (n\omega t) \quad \because \quad r^n = a^n \cos n\theta$

or $\quad r = a\,[\text{sech } (n\omega t)]^{1/n} \quad \ldots(4)$

Again normal equation of motion is

$$m\frac{v^2}{\rho} = R' - mr\omega^2 \sin\phi.$$

$$\therefore \quad R' = m\frac{v^2}{\rho} + mr\omega^2 \cos n\theta, \quad \because \quad \phi = \frac{\pi}{2} + n\theta.$$

Put $v^2 = \omega^2 r^2$ and $\rho = \dfrac{a^n}{n+1} \cdot \dfrac{1}{r^{n-1}}$ by (3) and (.

$$\therefore \quad R' = m \,.\, \omega^2 r^2 \,.\, \frac{(n+1)\, r^{n-1}}{a^n} + mr\omega^2 \,.\, \frac{r^n}{a^n}$$

$$= \frac{m\omega^2}{a^n}\, r^{n+1} \,.\, (n+2). \quad \ldots(5)$$

Now from (2) $R' = R - 2m\omega\,(-v) = R + 2m\omega v$.

$$\therefore \quad R = R' - 2m\omega v$$

or
$$R = \frac{m\omega^2}{a^n}\, r^{n+1}\,(n+2) - 2m\omega \,.\, (\omega r) \quad \because \quad v = \omega r$$

$$= \frac{m\omega^2 r}{a^n}\, [r^n\,(n+2) - 2a^n].$$

R will be zero when $r^n = \dfrac{2a^n}{n+2}$; $\therefore$ $r = a \,.\, \left(\dfrac{2}{n+2}\right)^{1/n}$ ...(6)

Now we have to find the time when $R=0$,

or
$$r = a\left(\frac{2}{n+2}\right)^{1/n}.$$

Also from (4), we have $r=a\,[\text{sech}\,(n\omega t)]^{1/n}$.

$$\therefore\quad a.\left(\frac{2}{n+2}\right)^{1/n}=a\,[\text{sech}\,(n\omega t)]^{1/2};$$

$$\therefore\quad \cosh\ (n\omega t)=\frac{n+2}{2}=1+\frac{n}{2}$$

or
$$1+2\sinh^2\frac{n\omega t}{2}=1+\frac{n}{2}.$$

$$\therefore\quad \sinh^2\frac{n\omega t}{2}=\frac{n}{4}\quad\text{or}\quad \sinh\frac{n\omega t}{2}=\frac{\sqrt{n}}{2}$$

or
$$\frac{n\omega t}{2}=\sinh^{-1}\left(\frac{\sqrt{n}}{2}\right).$$

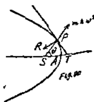
$$\therefore\quad t=\frac{2}{n\omega}\sinh^{-1}\left(\frac{\sqrt{n}}{2}\right).$$

Hence proved.

Ex. 6. *A particle can move in a smooth plane tube in the form of a parabola and the tube is initially at rest with the particle at vertex and then suddenly begins to rotate about the focus with uniform angular velocity. Prove that pressure in any position is proportional to* $\sqrt{r}\,(3\sqrt{a}-4\sqrt{r})$ *where* r *is the distance from the focus and* $4a$ *the latus rectum.*

(Rajputana 62 ; Agra 63)

Equation of the parabola is $\dfrac{2a}{r}=1+\cos\theta$,



$\phi=90-\dfrac{\theta}{2}$, pedal equation is $p^2=ar$,

or
$$p=\sqrt{a}\sqrt{r},\quad\therefore\quad \frac{dp}{dr}=\frac{\sqrt{a}}{2\sqrt{r}},$$

$$\rho=r.\frac{dr}{dp}=r.\frac{2\sqrt{r}}{\sqrt{a}}.$$

Proceeding exactly as in Q. 3 and 4, we have

$$mv\frac{dv}{ds}=mr\omega^2\cos\phi=mr\omega^2.\frac{dr}{ds}.$$

$$\therefore \quad v\,dv = \omega^2 r\,dr \quad \text{or} \quad v^2 = \omega^2 r^2 + A.$$

Initially when at the vertex $r = SA = a$, $v = a\omega$; $\therefore$ $A = 0$

$$\therefore \quad v^2 = \omega^2 r^2.$$

Again normal equation of motion is

$$\frac{mv^2}{\rho} = R' - mr\omega^2 \sin\phi.$$

$$\therefore \quad R' = \frac{m\omega^2 r^2}{2r\sqrt{\left(\frac{r}{a}\right)}} + mr\omega^2 . \frac{p}{r}$$

$$= \frac{m\omega^2}{2}\sqrt{(ar)} + m\omega^2\sqrt{(ar)} = \tfrac{3}{2}m\omega^2\sqrt{(ar)}.$$

Also if R be the true reaction when the curve is revolving, then $R' = R - 2m\omega v$, where v is the velocity of the particle relative to curve which is $-$ive here as both are in opposite directions. Since the particle is at rest at the vertex when the motion ensued, therefore its velocity relative to the curve should be in opposite direction.

$$\therefore \quad R' = R - 2m\omega(-v) = R + 2m\omega v \quad \text{or} \quad R = R' - 2m\omega v$$

or $$R = \frac{3}{2}\,m\omega^2\sqrt{(ar)} - 2m\omega\,.\,\omega r = \frac{m\omega^2}{2}[3\sqrt{(ar)} - 4r].$$

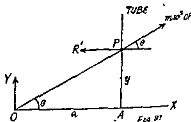
$\therefore$ R is proportional to $[3\sqrt{(ar)} - 4r]$

or $$\sqrt{r}\,(3\sqrt{a} - 4\sqrt{r}).$$

Ex. 7. *A particle is placed in a smooth straight tube which is suddenly set rotating with constant angular velocity ω about a point O in its plane of motion, which is at a perpendicular distance a from the tube. Show that the distance described along the tube by the particle in time t is a sinh (ωt), the particle and O being initially at a distance a apart. Prove also that the reaction between the tube and the particle is then* $\quad ma\omega^2\,(2\cosh\omega t - 1)$. **(Punjab 57)**

Let P be the position of the particle after time t moving along the tube, initially being at A, where $OA=a$.

In order to reduce the tube to rest we introduce a force $m\omega^2.\ OP$ along OP.



For all positions of P, $x=a$, y is changing.

Let R' be the reaction when the tube is taken to be at rest and R be the actual reaction when the tube is revolving, so that $R'=R-2m\omega v$, v being the velocity of the particle relative to tube. But since the direction of the tube and the particle will be opposite, therefore we take v to be $-$ive.

$$\therefore \quad R'=R-2m\omega\,(-v)=R+2m\omega v$$

or
$$R=R'-2m\omega. \qquad ...(1)$$

Here we write down the equation of motion along and perpendicular to the tube, *i e.* along the axes of coordinates.

$$m\frac{d^2x}{dt^2}=m\omega^2.OP\cos\theta-R'.$$

But x is constant and always equal to a; $\therefore \quad \frac{d^2x}{dt^2}=0.$

$$\therefore \quad R'=m\omega^2.OP\cos\theta=m\omega^2.OP.\frac{a}{OP}=m\omega^2a. \qquad ..(2)$$

Also since $\quad \frac{d^2x}{dt^2}=0, \quad \therefore \quad \frac{dx}{dt}=A.$

Since there is no motion along the axis of x,

$$\therefore \quad \frac{dx}{dt}=0. \qquad ...(3)$$

Again
$$m\frac{d^2y}{dt^2}=m\omega^2.OP\sin\theta$$

or
$$\frac{d^2y}{dt^2}=\omega^2.OP.\frac{y}{OP}=\omega^2.y.$$

Multiplying both sides by $2\frac{dy}{dt}$ and integrating, we get

$$\left(\frac{dy}{dt}\right)^2=\omega^2y^2+B.$$

Initially when $y=0$ at A, $\frac{dy}{dt}=a\omega$; $\therefore$ $B=a^2\omega^2$.

$$\therefore \left(\frac{dy}{dt}\right)^2=\omega^2(a^2+y^2).$$

$$\therefore \frac{dy}{dt}=\omega\sqrt{(a^2+y^2)} \qquad ...(4)$$

or $$\frac{dy}{\sqrt{(a^2+y^2)}}=\omega\, dt$$

or $$\sinh^{-1}\frac{y}{a}=\omega t+C.$$

Initially when $t=0$, $y=0$ at A ; $\therefore$ $C=0$.

$$\therefore y=a\sinh(\omega t). \qquad ..(5)$$

Again resultant velocity of particle is $\sqrt{(\dot{x}^2+\dot{y}^2)}$

or $v=\dot{y}=\omega\sqrt{(a^2+y^2)}$, $\because$ $\dot{x}=0$ by (3)

or $v=\omega\sqrt{(a^2+a^2\sinh^2\omega t)}=\omega a\cosh\omega t$ by (5). ...(6)

$\therefore$ $R=R'-2m\omega v$ from (1)

$=m\omega^2a-2m\omega.(\omega a\cosh\omega t)$ by (6) and (2)

$=m\omega^2a\,(1-2\cosh\omega t)$

$=-m\omega^2a\,(2\cosh\omega t-1)$.

The $-$ive sign indicates that the direction of R and hence of R' will be opposite to that marked in the figure.

Ex. 8. *A bead is at rest on an equiangular spiral of angle α at a distance a from the pole. The plane of the spiral is horizontal and the spiral is made to revolve abont a vertical line through its pole with uniform angular velocity ω Prove that the bead comes to a position of relative rest at a distance $a\cos\alpha$ from the pole and that the reaction of the curve is then*

$\frac{1}{3}m\omega^2 a \sin\alpha$. *Show also that, when the bead is again at its original distance from the pole, the reaction is*

$$m\omega^2 a \sin\alpha\,(3+\sin^2\alpha). \qquad \text{(Agra 53)}$$

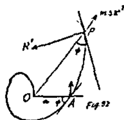
Polar equation of equi-angular spiral is $r=ae^{\theta\cot\alpha}$.

Taking log, $\log r=\log a+\theta\cot\alpha$.

Differentiating, $\frac{1}{r}\frac{dr}{d\theta}=\cot\alpha$ or $\cot\phi=\cot\alpha$; $\therefore\ \phi=\alpha$.

$p=r\sin\phi$; $\therefore\ p=r\sin\alpha$ is the pedal equation.

$$\frac{dp}{dr}=\sin\alpha; \qquad \therefore\ \rho=r\frac{dr}{dp}=\frac{r}{\sin\alpha}. \qquad \ldots(1)$$



In order to reduce the curve to rest we introduce a force $mr\omega^2$ along OP. R' is the normal reaction when the curve is taken to be at rest. The true reaction R is given by

$$R'=R-2m\omega v$$

or

$$R=R'+2m\omega v.$$

The tangential equation of motion is

$$mv\frac{dv}{ds}=mr\omega^2\cos\phi=mr\omega^2\cdot\frac{dr}{ds}.$$

$$\therefore\ v^2=\omega^2r^2+A.$$

Initially when the particle is at A, where $r=a$, it will describe a circle of radius a and hence its velocity will be $v=\frac{ds}{dt}=\frac{ds}{d\theta}\cdot\frac{d\theta}{dt}=a\omega$, along the tangent to the curve. Its component along the perpendicular to OA which will be tangent to the circle at A will be $a\omega\sin\phi=a\omega\sin\alpha$, $\because\ \phi=\alpha$.

Hence when $r=a$, $v=a\omega\sin\alpha$; $\therefore\ A=-a^2\omega^2\cos^2\alpha$.

$$\therefore\ v^2=\omega^2r^2-a^2\omega^2\cos^2\alpha. \qquad \ldots(2)$$

Now v will be zero, when $r=a\cos\alpha$.

Again the normal equation of motion is

$$m\frac{v^2}{\rho}=R'-m\omega^2r\sin\phi.$$

$\therefore \quad R'=m\dfrac{(\omega^2r^2-a^2\omega^2\cos^2\alpha)}{r\operatorname{cosec}\alpha}+m\omega^2 r\sin\alpha, \quad \because \quad \phi=\alpha.$ .. (3)

Now when $r=a\cos\alpha$, then

$$R'=0+m\omega^2(a\cos\alpha)\sin\alpha=\frac{m\omega^2 a}{2}\sin 2\alpha.$$

$\therefore$ from (2), $R=R'+2m\omega v$.

But at $r=a\cos\alpha$, $v=0$; $\therefore \quad R=R'=\dfrac{m\omega^2 a}{2}\sin 2\alpha.$

Again when the particle returns to the original position where $r=a$, then from (3),

$$R'=\frac{m\omega^2a^2(1-\cos^2\alpha)}{a\operatorname{cosec}\alpha}+m\omega^2 a\sin\alpha$$
$$=m\omega^2 a\sin^3\alpha+m\omega^2 a\sin\alpha.$$

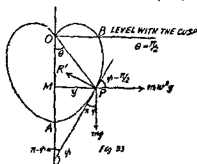
Also at $r=a$, $v^2=\omega^2a^2-a^2\omega^2\cos^2\alpha$. by (2)

$$=\omega^2a^2\sin^2\alpha$$

$\therefore \quad R=R'+2m\omega v$

$$=(m\omega^2 a\sin^3\alpha+m\omega^2 a\sin\alpha)+2m\omega(\omega a\sin\alpha)$$
$$=m\omega^2 a\sin\alpha(\sin^2\alpha+3).$$ **Proved.**

Ex. 9. *A tube in the form of the cardioid $r=a(1+\cos\theta)$ is placed with its axis vertical and cusp uppermost and revolves round the axis with angular velocity $\sqrt{\left(\frac{g}{a}\right)}$. A particle is projected from the lowest point of the tube along the tube with velocity $\sqrt{(3ga)}$. Show that the particle will ascend until it is on level with the cusp.* **(Raj. 62, Pb. 56, Agra 46, 49, 60, 65)**



Let P be the position of the particle at any time and its distance from the axis round which it revolves be y. In order to reduce the curve to rest we introduce a force $m\omega^2 y$ along MP. The tangent at P makes an angle ψ with axis of X and from the figure

it is clear that the direction of $m\omega^2 y$ will make an angle $\psi-\frac{\pi}{2}$ with the tangent, mg is the weight of the particle acting vertically downwards at P.

Tangential equation of motion is

$$mv\frac{dv}{ds}=m\omega^2 y.\cos\left(\psi-\frac{\pi}{2}\right)-mg\cos(\pi-\psi)$$

or $$v\frac{dv}{ds}=\omega^2 y\sin\psi+g\cos\psi.$$

Now $\tan\psi=\frac{dy}{dx}$, $\therefore$ $\sin\psi=\frac{dy}{ds}$, $\cos\psi=\frac{dx}{ds}$.

$$\therefore \quad v\frac{dv}{ds}=\omega^2 y.\frac{dy}{ds}+g\frac{dx}{ds}$$

or $$v\,dv=\omega^2 y\,dy+g\,dx.$$

Integrating, $$v^2=\omega^2 y^2+2gx+A.$$

Initially when the particle is at the vertex A, its cartesian co-ordinates are $(OA, 0)$ *i.e.*, $(2a, 0)$.

Hence when $x=2a$, $y=0$, $v=\sqrt{(3ga)}$. Also $\omega=\sqrt{\left(\frac{g}{a}\right)}$.

$$\therefore \quad 3ga=\frac{g}{a}.0+2g.2a+A; \qquad \therefore \quad A=-ga.$$

$$\therefore \quad v^2=\frac{g}{a}.y^2+2gx-ga.$$

Now v will be zero, when

$$ga=\frac{g}{a}y^2+2gx$$

or $a^2=r^2\sin^2\theta+2ar\cos\theta$, $\because$ $x=r\cos\theta$, $y=r\sin\theta$

or $a^2=a^2(1+\cos\theta)^2(1-\cos^2\theta)+2a.a(1+\cos\theta)\cos\theta$

or $1=(1+\cos\theta)^2-\cos^2\theta(1+\cos\theta)^2+2(1+\cos\theta)\cos\theta$

or $0=2\cos\theta+\cos^2\theta-\cos\theta(1+\cos\theta)[\cos\theta+\cos^2\theta-2]$

or $\quad 0=\cos\theta\,[2+\cos\theta-(\cos\theta+\cos^2\theta-2$
$\quad\quad -(\cos^2\theta+\cos$

or $\quad 0=\cos\theta\,[4-\cos^2\theta-\cos^2\theta-\cos^3\theta+2]$

or $\quad 0=\cos\theta\,[2\,(2+\cos\theta)-\cos^2\theta\,(2+\cos\theta)]$

or $\quad 0=\cos\theta\,.(2+\cos\theta)\,(2-\cos^2\theta)$

or $\quad 0=\cos\theta\,(2+\cos\theta)\,(\sqrt{2}-\cos\theta)\,(\sqrt{2}+\cos\theta).$

$\cos\theta=0$ is the only possible solution of above as all the other values of $\cos\theta$ given by above are greater than unity which are not possible, $\therefore\ \theta=\pi/2$.

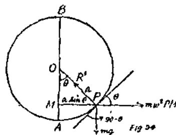
When $\theta=\frac{\pi}{2}$, the particle is at B, which is cleary in level with O. Hence the particle will ascend till it is in level with the cusp.

Ex. 10. *A circular ring of radius a revolves uniformly about a vertical diameter with angular velocity* $\sqrt{\left(\frac{ng}{a}\right)}$ *and a particle is projected from the lowest point with velocity just sufficient to carry it to the highest point : show that the time of describing the first quadrant is*

$$\sqrt{\left[\frac{a}{(n+1)\,g}\right]}\log\left[\sqrt{(n+2)}+\sqrt{(n+1)}\right].$$

(Vikram 63 ; Agra 45, 51, 54, 56)

Let P be the position of the particle at any time t and its distance from the vertical diameter about which it revolves be $PM=a\sin\theta$. In order to reduce the curve to rest, we introduce a force $m\omega^2\,PM$ along MP. The tangent at P is inclined at $90-\theta$



it is clear that the direction of $m\omega^2 y$ will make an angle $\psi-\frac{\pi}{2}$ with the tangent, mg is the weight of the particle acting vertically downwards at P.

Tangential equation of motion is

$$mv\frac{dv}{ds}=m\omega^2 y.\cos\left(\psi-\frac{\pi}{2}\right)-mg\cos(\pi-\psi)$$

or $$v\frac{dv}{ds}=\omega^2 y\sin\psi+g\cos\psi.$$

Now $\tan\psi=\frac{dy}{dx}$, $\therefore$ $\sin\psi=\frac{dy}{ds}$, $\cos\psi=\frac{dx}{ds}$.

$$\therefore\quad v\frac{dv}{ds}=\omega^2 y.\frac{dy}{ds}+g\frac{dx}{ds}$$

or $$v\,dv=\omega^2 y\,dy+g\,dx.$$

Integrating, $$v^2=\omega^2 y^2+2gx+A.$$

Initially when the particle is at the vertex A, its cartesian co-ordinates are $(OA, 0)$ *i.e.*, $(2a, 0)$.

Hence when $x=2a$, $y=0$, $v=\sqrt{(3ga)}$. Also $\omega=\sqrt{\left(\frac{g}{a}\right)}$.

$$\therefore\quad 3ga=\frac{g}{a}.0+2g.2a+A\,;\qquad\therefore\quad A=-ga.$$

$$\therefore\quad v^2=\frac{g}{a}.y^2+2gx-ga.$$

Now v will be zero, when

$$ga=\frac{g}{a}y^2+2gx$$

or $a^2=r^2\sin^2\theta+2ar\cos\theta$, $\because$ $x=r\cos\theta$, $y=r\sin\theta$

or $a^2=a^2(1+\cos\theta)^2(1-\cos^2\theta)+2a.a(1+\cos\theta)\cos\theta$

or $1=(1+\cos\theta)^2-\cos^2\theta(1+\cos\theta)^2+2(1+\cos\theta)\cos\theta$

or $0=2\cos\theta+\cos^2\theta-\cos\theta(1+\cos\theta)[\cos\theta+\cos^2\theta-2]$

$$\therefore \quad \frac{1}{2}\sqrt{\left(\frac{a}{g}\right)}\int_0^{\pi/2}\frac{d\theta}{\cos^2\frac{\theta}{2}\sqrt{\left(\sec^2\frac{\theta}{2}+n\tan^2\frac{\theta}{2}\right)}}$$

or
$$\frac{1}{2}\sqrt{\left(\frac{a}{g}\right)}\int_0^{\pi/2}\frac{\sec^2\frac{\theta}{2}\,d\theta}{\sqrt{\left\{1+(n+1)\tan^2\frac{\theta}{2}\right\}}}=T.$$

Put $\sqrt{(n+1)}\tan\frac{\theta}{2}=z$; $\therefore$ $\sqrt{(n+1)}\cdot\frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta=dz$.

Also when $\theta=0$, $z=0$; when $\theta=\frac{\pi}{2}$, then $z=\sqrt{(n+1)}$

$$\frac{1}{\sqrt{(n+1)}}\sqrt{\left(\frac{a}{g}\right)}\int_0^{\sqrt{(n+1)}}\frac{dz}{\sqrt{(1+z^2)}}=T$$

or
$$\sqrt{\left[\left\{\frac{a}{(n+1)\,g}\right\}\right]}\left[\log\{z+\sqrt{(1+z^2)}\}\right]_0^{\sqrt{(n+1)}}=T$$

or
$$\sqrt{\left\{\frac{a}{(n+1)\,g}\right\}}\log\{\sqrt{(n+1)}+\sqrt{(n+2)}\}=T.$$ **Proved.**

Ex. 11. *A particle P moves in a smooth circular tube of radius a which turns with uniform angular velocity ω about a vertical diameter. If the angular distance of the particle at any time t from the lowest point is θ and if it be at rest relative to the tube where $\theta=\alpha$ and $\cos\frac{\alpha}{2}=\frac{1}{\omega}\sqrt{\left(\frac{g}{a}\right)}$, then at any subsequent time t,*

$$\cot\frac{\theta}{2}=\cot\frac{\alpha}{2}\cosh\left(\omega\sin\frac{\alpha}{2}\right).$$

(Vikram 64 ; Agra 61 ; Punjab 60)

With the same figure as of Ex. 10 and arguing as before, the tangential equation of motion is

$$mv\frac{dv}{ds}=(m\omega^2 . PM)\cos\theta-mg\sin\theta$$

to the vertical. The weight mg of the particle acts vertically downwards at P.

Tangential equation of motion is

$$mv.\frac{dv}{ds}=(m\omega^2\ PM)\cos\theta-mg\sin\theta$$

or
$$v\frac{dv}{d\theta}\cdot\frac{d\theta}{ds}=\omega^2.a\sin\theta\cos\theta-g\sin\theta.$$

Now $\quad s=a\theta\ ;\ \therefore\ \frac{ds}{d\theta}=a,\omega=\sqrt{\left(\frac{ng}{a}\right)}$.

$$\therefore\ v.dv=a\left(\frac{ng}{a}\ a\sin\theta\cos\theta-g\sin\theta\right)d\theta.$$

$$\therefore\ v^2=a\ (ng\sin^2\theta+2g\cos\theta)+A.$$

We are given that velocity of projection is sufficient enough to carry the particle to highest point.

$\therefore\ v=0$, when $\theta=\pi$

$\therefore\ 0=a\ [0+2g\ (-1)]+A\ ;\ \therefore\ A=2ag.$

$$\therefore\ v^2=a\ [ng\sin^2\theta+2g\ (1+\cos\theta)]$$

$$=a\left[4ng\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}+4g\cos^2\frac{\theta}{2}\right].$$

$$\therefore\ v=+2\sqrt{(ag)}\cos\frac{\theta}{2}\sqrt{\left(1+n\sin^2\frac{\theta}{2}\right)}$$

[+ ive sign is due to the fact that θ is increasing with t and $v=a\frac{d\theta}{dt}=+$ ive].

$$a\frac{d\theta}{dt}=2\sqrt{(ag)}\cos\frac{\theta}{2}\sqrt{\left(1+n\sin^2\frac{\theta}{2}\right)}.$$

$$\therefore\ \frac{1}{2}\sqrt{\left(\frac{a}{g}\right)}\int_0^{\pi/2}\frac{1}{\cos\frac{\theta}{2}\sqrt{\left(1+n\sin^2\frac{\theta}{2}\right)}}\,d\theta=\int_0^T dt.$$

Initially when $t=0$, $\theta=0$ at the lowest point and when $t=T$ the particle has described a quadrant, *i. e.* $\theta=\pi/2$.

$$\therefore \quad \frac{1}{2}\sqrt{\left(\frac{a}{g}\right)}\int_0^{\pi/2}\frac{d\theta}{\cos^2\frac{\theta}{2}\sqrt{\left(\sec^2\frac{\theta}{2}+n\tan^2\frac{\theta}{2}\right)}}$$

or
$$\frac{1}{2}\sqrt{\left(\frac{a}{g}\right)}\int_0^{\pi/2}\frac{\sec^2\frac{\theta}{2}\,d\theta}{\sqrt{\left\{1+(n+1)\tan^2\frac{\theta}{2}\right\}}}=T.$$

Put $\sqrt{(n+1)}\tan\frac{\theta}{2}=z$; $\therefore \quad \sqrt{(n+1)}\cdot\frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta=dz.$

Also when $\theta=0$, $z=0$; when $\theta=\frac{\pi}{2}$, then $z=\sqrt{(n+1)}$

$$\frac{1}{\sqrt{(n+1)}}\sqrt{\left(\frac{a}{g}\right)}\int_0^{\sqrt{(n+1)}}\frac{dz}{\sqrt{(1+z^2)}}=T$$

or
$$\sqrt{\left[\left\{\frac{a}{(n+1)\,g}\right\}\right]}\left[\log\{z+\sqrt{(1+z^2)}\}\right]_0^{\sqrt{(n+1)}}=T$$

or
$$\sqrt{\left\{\frac{a}{(n+1)\,g}\right\}}\log\{\sqrt{(n+1)}+\sqrt{(n+2)}\}=T. \quad \textbf{Proved.}$$

Ex. 11. *A particle P moves in a smooth circular tube of radius a which turns with uniform angular velocity ω about a vertical diameter. If the angular distance of the particle at any time t from the lowest point is θ and if it be at rest relative to the tube where θ=α and* $\cos\frac{\alpha}{2}=\frac{1}{\omega}\sqrt{\left(\frac{g}{a}\right)}$, *then at any subsequent time t,*

$$\cot\frac{\theta}{2}=\cot\frac{\alpha}{2}\cosh\left(\omega\sin\frac{\alpha}{2}\right).$$

(Vikram 64 ; Agra 61 ; Punjab 60)

With the same figure as of Ex. 10 and arguing as before, the tangential equation of motion is

$$mv\frac{dv}{ds}=(m\omega^2 . PM)\cos\theta-mg\sin\theta$$

or $$v \frac{dv}{d\theta} \cdot \frac{d\theta}{ds} = (\omega^2 a \sin\theta \cos\theta - g \sin\theta)$$

or $$v\, dv = a\, [\omega^2 a \sin\theta \cos\theta - g \sin\theta]\, d\theta.$$

Integrating, we get

$$v^2 = 2a \left[a\omega^2 \frac{\sin^2\theta}{2} + g\cos\theta \right] + A. \qquad ...(1)$$

We are given that when $\theta = \alpha$, $v = 0$.

$$\therefore \quad 0 = 2a \left[a\omega^2 \frac{\sin^2\alpha}{2} + g\cos\alpha \right] + A. \qquad ...(2)$$

Subtracting (1) and (2), we get

$$v^2 = a^2\omega^2 (\sin^2\theta - \sin^2\alpha) + 2ag(\cos\theta - \cos\alpha). \qquad ...(3)$$

We are given that $\cos\frac{\alpha}{2} = \frac{1}{\omega}\sqrt{\left(\frac{g}{a}\right)}$; $\therefore \quad g = a\omega^2 \cos^2\frac{\alpha}{a}$.

Putting for g in (3) and $v = a\frac{d\theta}{dt}$, we get

$$a^2 \left(\frac{d\theta}{dt}\right)^2 = a^2\omega^2 (\sin^2\theta - \sin^2\alpha) + 2a^2\omega^2 \cos^2\frac{\alpha}{2} (\cos\theta - \cos\alpha).$$

$$\therefore \quad \left(\frac{d\theta}{dt}\right)^2 = \omega^2 \left[4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} - 4\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2} + 2\cos^2\frac{\alpha}{2}\left\{\left(1 - 2\sin^2\frac{\theta}{2}\right) - \left(1 - 2\sin^2\frac{\alpha}{2}\right)\right\}\right]$$

or $$\left(\frac{d\theta}{dt}\right)^2 = 4\omega^2 \left[\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} - \sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\cos^2\frac{\alpha}{2} \right]$$

$$= 4\omega^2 \left[\cos^2\frac{\theta}{2} - \cos^2\frac{\alpha}{2} \right] \sin^2\frac{\theta}{2}.$$

$$\therefore \quad \frac{d\theta}{dt} = -2\omega \left[\cos^2\frac{\theta}{2} - \cos^2\frac{\alpha}{2} \right]^{1/2} \sin\frac{\theta}{2}.$$

Here we have chosen $-$ive sign and not $+$ive as in last question, the reason being that the particle did not start when

$\theta=0$. It started from $\theta=\alpha$ and now θ is decreasing as t is increasing and hence $\frac{d\theta}{dt}$ is $-$ve.

$$\therefore \quad \int_{\alpha}^{0} \frac{d\theta}{-2\omega \sin \frac{\theta}{2}\sqrt{\left(\cos^2 \frac{\theta}{2}-\cos^2 \frac{\alpha}{2}\right)}}=\int_0^t dt.$$

$t=0$ corresponds to the position of start where $\theta=\alpha$ given and $t=t$ corresponds to the position where $\theta=\theta$.

$$\therefore \quad \frac{1}{2\omega}\int_{\alpha}^{0} \frac{-d\theta}{\sin \frac{\theta}{2} \sin \frac{\theta}{2}\sqrt{\left(\cot^2 \frac{\theta}{2}-\cos^2 \frac{\alpha}{2} \operatorname{cosec}^2 \frac{\theta}{2}\right)}}=\Big[\, t \,\Big]_0^t$$

Put $\operatorname{cosec}^2 \frac{\theta}{2}=1+\cot^2 \frac{\theta}{2}$

$$\therefore \quad \frac{1}{2\omega}\int_{\alpha}^{0} \frac{-\operatorname{cosec}^2 \frac{\theta}{2}\, d\theta}{\sqrt{\left[\cot^2 \frac{\theta}{2}\left(1-\cos^2 \frac{\alpha}{2}\right)-\cos^2 \frac{\alpha}{2}\right]}}=t$$

$$\frac{1}{2\omega}\int_{\alpha}^{\theta} \frac{-\operatorname{cosec}^2 \frac{\theta}{2}\, d\theta}{\sin \frac{\alpha}{2}\sqrt{\left(\cot^2 \frac{\theta}{2}-\cot^2 \frac{\alpha}{2}\right)}}=t$$

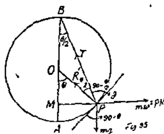
$$=\frac{1}{\omega \sin \frac{\alpha}{2}}\left\{\cosh^{-1} \frac{\cot \frac{\theta}{2}}{\cot \frac{\alpha}{2}}\right\}_{\alpha}^{\theta}=t$$

$$\therefore \quad \cosh^{-1} \frac{\cot \frac{\theta}{2}}{\cot \frac{\alpha}{2}}-\cosh^{-1} 1=\left(\omega t \sin \frac{\alpha}{2}\right).$$

$$\therefore \quad \cot \frac{\theta}{2}=\cot \frac{\alpha}{2} \cosh \left(\omega t \sin \frac{\alpha}{2}\right) \quad \because \cosh^{-1} 1=0.$$

Ex. 12. *A thin circular wire is made to revolve about a vertical diameter with constant angular velocity. A smooth ring slides on the wire, being attached to its highest point by an elastic string whose natural length is equal to radius of wire. If the ring be slightly displaced from the lowest point find the motion and show that it will reach the highest point if the modulus of elasticity is equal to four times the weight of the particle* (Agra 47)

All the forces are the same as in Ex. 9, 10, 11 except that one more force *i. e.* tension T is also introduced. The force $m\omega^2\ PM = m\omega^2 a \sin\theta$ is applied in order to reduce the curve to rest.



When the ring is at P, radius to which makes an angle θ with the vertical, then, $\angle PBO = \frac{\theta}{2}$,

as angle subtended at centre is double the angle subtended by the same arc at any point on the circumference. Natural length of the string is a given and extended length

$= BP = AB \cos\frac{\theta}{2} = 2a\cos\frac{\theta}{2}$ because $\angle APB = \frac{\pi}{2}$.

$$\therefore \quad \text{Extension} = \left(2a\cos\frac{\theta}{2} - a\right).$$

$$\therefore \quad T = \lambda . \frac{\text{Extension}}{\text{Natural length}} = \lambda . \frac{a\left(2\cos\frac{\theta}{2} - 1\right)}{a}.$$

or
$$T = \lambda\left(2\cos\frac{\theta}{2} - 1\right) \qquad \text{.. (1)}$$

Tangential equation of motion as in Ex. 9, 10, 11 is

$$mv\frac{dv}{ds} = (m\omega^2 a \sin\theta)\cos\theta - mg\sin\theta + T\sin\frac{\theta}{2}.$$

or $\quad v\dfrac{dv}{d\theta}\cdot\dfrac{d\theta}{ds}=\omega^2 a\sin\theta\cos\theta-g\sin\theta\,\dfrac{\lambda}{m}\left(2\cos\dfrac{\theta}{2}-1\right)\sin\dfrac{\theta}{2}.$

$$\therefore\ v\,dv=a\left[a\omega^2\sin\theta\cos\theta-g\sin\theta+\frac{\lambda}{m}\left(\sin\theta-\sin\frac{\theta}{2}\right)\right]d\theta$$

$$\therefore\quad v^2=2a\left[a\omega^2\frac{\sin^2\theta}{g}+g\cos\theta\right.$$

$$\left.+\frac{\lambda}{m}\left(-\cos\theta+2\cos\frac{\theta}{2}\right)+A.\right. \quad ..(2)$$

Initially when $\theta=0$, $v=0$.

$$\therefore\quad 0=2a\left[g+\frac{\lambda}{m}(-1+2)\right]+A$$

$$\therefore\quad A=-2ag-2\,\frac{a\lambda}{m}.$$

$$\therefore\quad v^2=2a\left[a\omega^2\frac{\sin^2\theta}{2}+g\cos\theta\right.$$

$$\left.+\frac{\lambda}{m}\left(-\cos\theta+2\cos\frac{\theta}{2}\right)\right]-2ag-\frac{2a\lambda}{m}. \quad ...(3)$$

The above equation of motion will hold good so long as there is tension in the string. Now tension will be zero if by (1), $\cos\dfrac{\theta}{2}=\dfrac{1}{2}=\cos\dfrac{\pi}{3}$; $\therefore\quad \theta=\dfrac{2\pi}{3}=120^\circ$

Also the velocity at this instant is obtained by putting $\theta=\dfrac{2\pi}{3}$ in (3) *i.e.* $\sin\theta=\dfrac{\sqrt{3}}{2}$ and $\cos\theta=-\dfrac{1}{2}$.

$$\therefore\ v^2=2a\left[\frac{a\omega^2}{2}\left(\frac{\sqrt{3}}{2}\right)^2+g\left(-\frac{1}{2}\right)+\frac{\lambda}{m}\left(\frac{1}{2}+2\cdot\tfrac{1}{2}\right)\right]-2ag-\frac{2a\lambda}{m}$$

or $$v^2=\frac{3}{4}a^2\omega^2-ag+3\,\frac{a\lambda}{m}-2ag-\frac{2a\lambda}{m}$$

or $$v^2=\frac{3}{4}a^2\omega^2-3ag+\frac{a\lambda}{m}. \quad ...(4)$$

From this position onwards the motion will be without

T. Hence putting $\lambda=0$ in (2) and proceeding as before, we have $v^2=2a\left[a\omega^2\frac{\sin^2\theta}{2}+g\cos\theta\right]+B.$ ...(5)

At the beginning of the motion $\theta=\frac{2\pi}{3}$ and v^2 is given by (4).

$$\therefore \quad \frac{3}{4}a^2\omega^2-3ag+\frac{a\lambda}{m}=2a\left[\frac{a\omega^2}{3}\cdot\frac{3}{4}+g\left(-\frac{1}{2}\right)\right]+B.$$

$$\therefore \quad -2ag+\frac{a\lambda}{m}=B.$$

Putting for B in (5), we get the velocity of the particle beyond the position $\theta=\frac{2\pi}{3}$ as

$$v^2=2a\left[a\omega^2\frac{\sin^2\theta}{2}+g\cos\theta\right]-2ag+\frac{a\lambda}{m}.$$

In order that the particle may reach the highest point, we put $v=0$ when $\theta=\pi$.

$$\therefore \quad 0=2a\,[0+g\,(-1)]-2ag+\frac{a\lambda}{m}$$

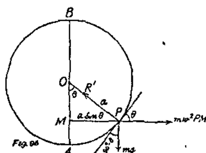
or $$0=-4ag+\frac{a\lambda}{m}\,; \quad \therefore \quad \lambda=4mg$$

or Modulus of elasticity = four times the weight of the particle.

Ex. 13. *A small bead slides on a circular arc of radius a which revolves with constant angular velocity ω about its vertical diameter. Find the position of stable equilibrum according as $\omega^2 >$ or $< \frac{g}{a}$ and show that the time of a small oscillation about its position of equilibrium is for two cases respectively equal to*

$$\frac{2\pi\omega a}{\sqrt{(a^2\omega^4-g^2)}} \text{ and } \frac{2\pi}{\sqrt{\left(\frac{g}{a}-\omega^2\right)}}$$

We introduce a force $m\omega^2\, PM = m\omega^2 a \sin\theta$ along MP to reduce the revolving curve at rest and the tangential equation of motion as usual is



$$m\frac{d^2s}{dt^2} = -mg\sin\theta$$

$$+(m\omega^2 a \sin\theta)\cos\theta.$$

But $s = a\theta$; $\therefore \quad \frac{ds}{dt} = a\frac{d\theta}{dt}$ and $\frac{d^2s}{dt^2} = a\frac{d^2\theta}{dt^2}$

or $$a\frac{d^2\theta}{dt^2} = -g\sin\theta + \omega^2 a\sin\theta\cos\theta. \quad \text{...(1)}$$

For stable equilibrium, we have $\frac{d^2\theta}{dt^2} = 0$.

$$\therefore \quad \sin\theta\,(\omega^2 a\cos\theta - g) = 0. \quad \text{...(2)}$$

Above gives us either $\sin\theta = 0$ or $\cos\theta = \frac{g}{a\omega^2}$. ...(3)

Now if $\omega^2 < \frac{g}{a}$, then $\frac{g}{a\omega^2} > 1$ *i.e.* $\cos\theta > 1$

which is not possible, and hence we shall choose

$$\sin\theta = 0 \quad \text{or} \quad \theta = 0.$$

Here for small oscillations about the position of equilibrium we choose $\theta = 0 + \phi$ where ϕ is very small.

$$\therefore \quad \frac{d^2\theta}{dt^2} = \frac{d^2\phi}{dt^2}.$$

Putting in (1), we get

$$a\frac{d^2\phi}{dt^2} = -g\sin\phi + \omega^2 a\sin\phi\cos\phi$$

$$= -g\left(\phi - \frac{\phi^3}{3!} - \right) + \omega^2 a\left(\phi - \frac{\phi^3}{3!}\right)\left(1 - \frac{\phi^2}{2!}\cdots\right)$$

$= -g\phi + a\omega^2\phi$, neglecting higher powers of ϕ, which is small.

$$\therefore \quad \frac{d^2\phi}{dt^2}=-\left(\frac{g}{a}-\omega^2\right)\phi.$$

$\therefore$ we have taken $\omega^2<\frac{g}{a}$, *i.e.* $\frac{g}{a}>\omega^2$.

Above is of the form $\frac{d^2x}{dt^2}=-\mu x$, *i. e.* S. H. M. and its periodic time is $\frac{2\pi}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{\left(\frac{g}{a}-\omega^2\right)}}$.

In the other case when $\omega^2>\frac{g}{a}$, *i.e.* $\frac{g}{a\omega^2}<1$, then the value of θ given by $\cos\theta=\frac{g}{a\omega^2}<1$, from (2) is a possibility. Let us suppose that $\frac{g}{a\omega^2}=\cos\alpha$. ...(4)

$$\therefore \quad \cos\theta=\cos\alpha, \quad i.e. \quad \theta=\alpha.$$

Here for small oscillations about the position of equilibrium we have $\theta=\alpha+\psi$ where ψ is small.

$$\therefore \quad \frac{d^2\theta}{dt^2}=\frac{d^2\psi}{dt^2}.$$

Hence from (1), we have

$$a\frac{d^2\psi}{dt^2}=-g\sin(\alpha+\psi)+\frac{a\omega^2}{2}\sin 2(\alpha+\psi)$$

$$=-g(\sin\alpha\cos\psi+\cos\alpha\sin\psi)+\frac{a\omega^2}{2}(\sin 2\alpha\cos 2\psi+\cos 2\alpha\sin 2\psi)$$

$$=-g(\sin\alpha.\,1+\cos\alpha.\,\psi)+\frac{a\omega^2}{2}(\sin 2\alpha.\,1+2\psi\cos 2\alpha),$$

$\therefore$ when ψ is small, $\sin\psi=\psi$ and $\cos\psi=1$ approx.

or $$a\frac{d^2\psi}{dt^2}=\left(-g\sin\alpha+\frac{a\omega^2}{2}.2\sin\alpha\cos\alpha\right)+[-g\cos\alpha+a\omega^2(2\cos^2\alpha-1)]\psi.$$

Put $\cos\alpha=\frac{g}{a\omega^2}$ from (4).

$$a\frac{d^2\psi}{dt^2}=\left(-g\sin\alpha+a\omega^2\sin\alpha\ \frac{g}{a\omega^2}\right)$$
$$+\left\{-\frac{g^2}{a\omega^2}+a\omega^2\left(2\frac{g^2}{a^2\omega^4}-1\right)\right\}\psi$$

or
$$a\frac{d^2\psi}{dt^2}=\left\{\frac{g^2}{a\omega^2}-a\omega^2\right\}\psi=-\frac{a^2\omega^4-g^2}{a\omega^2}\psi.$$

We have written as above because $\omega^2>\frac{g}{a}$,

$\therefore\ a^2\omega^4-g^2$ is +ive.

$$\therefore\ \frac{d^2\psi}{dt^2}=-\frac{a^2\omega^4-g^2}{a^2\omega^4}\psi.$$

Above is S. H. M. and periodic time is $\frac{2\pi a\omega}{\sqrt{(a^2\omega^4-g^2)}}$.

Ex. 14. *A parabolic wire whose axis is vertical and vertex downwards, rotates about its axis with uniform angular velocity* ω. *A ring is placed at any point of it in relative rest. Show that it will move upwards or downwards according as* $\omega^2>$ *or* $<\frac{g}{2a}$ *and will remain at rest, if* $\omega^2=\frac{g}{2a}$, *where* $4a$ *is the latus rectum of the parabola.* **(Vikram 65 ; Pb. 56)**

The equation of the parabola is $x^2=4ay$. In order to reduce the curve to rest we introduce a force $m\omega^2.PM$,

i.e. $m\omega^2x$ along MP.

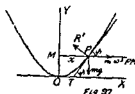


Fig 97

The ring is in relative rest at P. Hence it will move upwards, remain at rest or downwards according as its tangential acceleration is +ive, zero or −ive.

The tangential equation of motion is

$$mv\frac{dv}{ds}=m\omega^2 x\cos\psi-mg\sin\psi$$

or
$$v\frac{dv}{ds}=\omega^2 x\cos\psi-g\sin\psi. \quad \ldots(1)$$

From the given equation, $\dfrac{dy}{dx}=\dfrac{2x}{4a}=\dfrac{x}{2a}=\tan\psi.$

$$\therefore \quad \sin\psi=\frac{x}{\sqrt{(x^2+4a^2)}},\ \cos\psi=\frac{2a}{\sqrt{(x^2+4a^2)}}.$$

Putting for $\sin\psi$ and $\cos\psi$ in (1), we get

$$v\frac{dv}{ds}=\omega^2 x.\frac{2a}{\sqrt{(x^2+4a^2)}}-g.\frac{x}{\sqrt{(x^2+4a^2)}}$$

$$=\frac{2ax}{\sqrt{(x^2+4a^2)}}\left[\omega^2-\frac{g}{2a}\right].$$

Now tangential acceleration will be +ive, zero or −ive according as $\qquad \omega^2 >=< \dfrac{g}{2a}.$

Ex. 15. *A smooth circular wire rotates with uniform angular velocity ω about its tangent line at a point A. A bead, without weight, slides on the wire from a position of rest at a point of the wire very near A. Show that the angular distance on the wire traversed in time t after passing the point opposite A is* $\tan^{-1}(\omega t)$.

When the particle is at P after crossing B the point opposite to A after travelling an angular distance θ,

i.e., $\qquad \angle AOP=\theta.$

$\therefore \quad \angle BOP=\angle AOP-\angle AOB$

or $\quad \angle BOP=\theta-\pi=\angle MPO.$

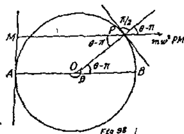


Fig 98

We are to find the time from B to P or during which θ changes from π to θ and the angular distance is $\theta-\pi$ whose value we are to find.

Introduce a force $m\omega^2\, PM$ along PMd in orer to reduce the curve to rest.

Also $\quad PM = a + a\cos(\theta - \pi) = a(1 - \cos\theta)$.

Since the bead is without weight, as such there will be no force like mg as in other questions.

The direction of $m\omega^2\, PM$ makes with the tangent an angle $\quad \theta - \pi + \frac{\pi}{2} = \theta - \frac{\pi}{2} = -\left(\frac{\pi}{2} - \theta\right)$.

Hence the component of $m\omega^2.\, PM$ along the tangent is

$$m\omega^2.PM\cos\left\{-\left(\frac{\pi}{2}-\theta\right)\right\} = m\omega^2.PM\cos\left(\frac{\pi}{2}-\theta\right)$$

$$= m\omega^2.PM\sin\theta.$$

$\therefore$ Tangential equation of motion is

$$mv\frac{dv}{ds} = m\omega^2.PM\sin\theta$$

or $\quad v\frac{ds}{d\theta}\cdot\frac{d\theta}{ds} = \omega^2 a(1-\cos\theta)\sin\theta, \quad \because PM = a(1-\cos\theta).$

But $\frac{ds}{d\theta} = a$ in a circle; $\quad \therefore \quad \frac{d\theta}{ds} = \frac{1}{a}$.

$$\therefore \quad v\,dv = \omega^2 a^2(1-\cos\theta)\sin\theta . d\theta.$$

Integrating, $\quad v^2 = \omega^2 a^2 (1-\cos\theta)^2 + A$.

Initially when $\theta = 0$, *i.e.* at A, $v = 0$; $\quad \therefore \quad A = 0$.

$$\therefore \quad a^2\left(\frac{d\theta}{dt}\right)^2 = \omega^2 a^2(1-\cos\theta)^2$$

or $$\frac{d\theta}{dt} = \omega(1-\cos\theta) = \omega . 2\sin^2\frac{\theta}{2}.$$

$$\therefore \int_{\pi}^{\theta} \frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}\,d\theta = \int_{t_1}^{t_2}\omega\,dt.$$

$$\therefore \; -\left[\cot\frac{\theta}{2}\right]_{\pi}^{\theta} = \omega(t_2 - t_1)$$

or $$-\left[\cot\frac{\theta}{2} - \cot\frac{\pi}{2}\right] = \omega T,$$

where V is the time from B to P

or $$-\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)=\omega T$$

or $$\tan\left(\frac{\theta}{2}-\frac{\pi}{2}\right)=\omega T$$

or $$\frac{\theta-\pi}{2}=\tan^{-1}\omega T$$

or $$\theta-\pi=2\tan^{-1}\omega T.$$

From the figure it is clear that in moving from B to P the angular distance traversed is $\theta-\pi$.

$\therefore$ Angular distance $=2\tan^{-1}(\omega T)$.

We may now replace T by t.

Ex. 16. *A bead can move on a smooth circular wire and is initially at rest at a point A. The wire is made to rotate uniformly on its own plane with angular velocity ω about the other end of the diameter through O. Show that the pressure vanishes after a time $\frac{1}{\omega}\log\{\frac{1}{2}(3+\sqrt{5})\}$ and that the angle described by the bead about the centre of the wire in time t is $4\tan^{-1}(\tanh\frac{1}{2}\omega t)$.*

Ex. 17. *A particle P is free to move on a smooth circular wire whose centre C revolves with constant angular velocity in the plane of the wire about a fixed point O. Show that if $CP=3CO$ and the particle just makes complete revolution, the pressure vanishes when CP makes with OC an angle $\sec^{-1}3$.*



OSMANIA UNIVERSITY PAPERS
1966

1 (*a*) If a particle, under a central force, describes an orbit in a plane, prove, with the usual notation, that its differential equation is

$$\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}.$$

(*b*) If the resistance offered by the medium on a body is equal to K times its velocity per unit mass, prove that the differential equation of the orbit, under the central acceleration $-P$, is

$$\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}e^{2Kt},$$ with the usual notation.

2. (*a*) A particle fall from rest under gravity in a medium, which offers a resistance varying as the square of its velocity. Discuss the motion.

(*b*) A particle is projected vertically upwards with a velocity u in a medium whose resistance is Kv^4 per unit mass, v being the velocity, show that the greatest height reached is

$$\frac{1}{2\sqrt{(Kg)}}\tan^{-1}\left\{\sqrt{\left(\frac{K}{g}\right)}\,u^2\right\}.$$

DELHI UNIVERSITY (Hons.) PAPERS
1963

1. A particle moves with a central acceleration $\frac{\mu}{(\text{distance})^3}$; find the path and distinguish between the different cases.

A particle moves with a central acceleration which varies inversely as the cube of the distance; if it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, shew that the equation to its path is $r \cos \frac{\theta}{\sqrt{2}}=a$.

2. A particle while describing an elliptic orbit receives an impulse and describes another path. Find expressons for the change (in magnitude) in the major axis and the eccentricity of the orbit.

A planet, of mass M and periodic time T, when at its greatest distance from the sun comes into collison with a meteor of mass m, moving in the same orbit in the opposite direction with velocity v; if $\frac{m}{M}$ be small shew that the major axis of the planet's path is reduced by

$$\frac{4m}{M} \cdot \frac{vT}{\pi} \sqrt{\left(\frac{1-e}{1+e}\right)}.$$

3. (*a*) A particle is moving downwards on a rough cycloid whose axis is vertical and vertex downward, the coefficient of friction being μ. Find the motion.

(*b*) A small bead moves on a thin elliptic wire under a force to the focus equal to $\frac{\mu}{r^2}+\frac{\lambda}{r^3}$. It is projected from a point on the wire distant R from the focus with the velocity

which would cause it to describe the ellipse freely under a force $\frac{\mu}{r^2}$. *Shew that the reaction of the wire is*

$$\frac{\lambda}{\rho}\left[\frac{1}{r^2}-\frac{1}{ar}+\frac{1}{R^2}\right],$$

where ρ is the radius of curvature.

1964

1. (*a*) A particle moves in an ellipse under a force which is always directed towards its focus; find the law of force and the velocity at any point of its path.

(*b*) A particle moves with a central acceleration $\mu(r^5-c^4r)$, being projected from an apse at distance c with a velocity $\sqrt{\left(\frac{2\mu}{3}\right)}\,c^3$; shew that its path is the curve

$$x^4+y^4=c^4.$$

2. (*a*) A particle while describing an elliptic orbit under a central force receives an impulse and describes another path. Find expressions for the changes (in magnitude) in the major axis and the eccentricity of the orbit.

(*b*) Two particles, of masses m_1 and m_2, moving in coplanar parabolas round the Sun, collide at right angles and coalesce when their common distance from the sun is R. Shew that the subsequent path of the combined particles is an ellipse of major axis $\frac{(m_1+m_2)^2}{2m_1m_2}R$.

—o—

1965

1. (*a*) A smooth cycloidal curve, with axis vertical and vertex downwards, is in a vertical plane. A particle is constrained to move along it, under gravity. Discuss the motion.

(*b*) A bead, which slides on a smooth wire in the form of a parabola with its axis vertical and vertex upwards, is just displaced from rest at the highest point. Show that in any subsequent position the velocity of the bead varies as its distance from the axis of the parabola ; also find the pressure on the curve.

2. (*a*) Find the path of a particle moving under the action of a force to a fixed point, varying inversely as the square of the distance from the fixed point.

(*b*) A particle of mass m is projected under the force $\frac{m\mu}{r^2}$ directed towards a fixed centre. If it be projected from the point $r=c$, $\theta=0$, at right angles to the radius with a velocity $\sqrt{\frac{n\mu}{c}}$, prove that

$$\left\{r\left(\frac{dr}{dt}\right)\right\}^2=\frac{\mu}{c}(r-c)\{nc-(2-n)r\}.$$

CALCUTTA UNIVERSITY (Hons.) PAPERS

1963

1. A particle describes an ellipse under a force varying inversely as the square of the distance from a focus. Show that the square of the time of describing the ellipse varies as the cube of the major axis.

A particle under a force $\mu/(\text{distance})^2$ tending to a fixed points is projected with velocity V at a distance R from the centre of force. Prove that if the path is a rectangular hyperbola, then the angle of projection is

$$\sin^{-1} \frac{\mu}{VR\left[V^2-\frac{2\mu}{R}\right]^{1/2}}.$$

2. (*a*) A particle is moving in a straight line with an acceleration μx towards a fixed centre in the straight line with an additional acceleration $L \cos pt$, where μ, L, p are constants. If the particle starts from rest at a distance a at zero time, find the displacement at time t.

Discuss also the case when $p \to \sqrt{(\mu)}$

(*b*) If V be the terminal velocity for a particle of mass m falling against a resistance proportional to the square of the velocity, prove that the kinetic energy acquired in falling through h feet from rest is

$$\tfrac{1}{2}mV^2\left(1-e^{-\frac{2gh}{V^2}}\right).$$

1965

1. Prove that the force per unit mass f, directed to the origin, under which a central orbit is described is given by the formula,

$$f=\frac{h^2}{p^3}\frac{dp}{dr},$$

where p is the length of the perpendicular from the origin on the tangent and h is the constant moment of the velocity about the origin.

Find the law of force to the pole when the path is the cardioid $r=a\,(1-\cos\theta)$, and prove that, if F be the force at the apse, and v the velocity,

$$3v^2=4aF.$$

2. A particle describes a path which is nearly a circle about a centre of force $\mu\phi(u)$. Find the condition that this may be a stable motion. If the radius is c^{-1} and $u=r^{-1}$, show that in this case the apsidal angle is

$$\pi\bigg/\left\{3-\frac{c\phi'(c)}{\phi(c)}\right\}^{1/2}.$$

Two particles are projected simultaneously from O in different directions with the same speed u so as to pass through another point P. If α, β are the angles of projection prove that they pass through P at times separated by

$$\frac{2u}{g}\sin\tfrac{1}{2}(\alpha-\beta)\sec\tfrac{1}{2}(\alpha+\beta).$$

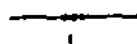
3. A particle falls from rest under gravity in a medium whose resistance is k (velocity)^2; the terminal speed c is $\sqrt{(g/k)}$; show that at any time t if v be the velocity and x distance descended :

(*i*) $v=c\tanh\dfrac{gt}{c}$

(*ii*) $x=\dfrac{c^2}{g}\log\cosh\dfrac{gt}{c}$.

A body of mass M lbs. is propelled in a straight line by a force which works at constant horse-power H against a resistance kv^2 lbs. wt., where v is the speed in feet per second. Prove that the distance described from rest until the speed is v is

$$\frac{M}{3kg}\log\frac{550H}{550H-kv^3}\text{ feet.}$$



SAGAR UNIVERSITY PAPERS

1963

1. A particle is projected under gravity and a resistance equal to mk (velocity) with a velocity u at an angle α to the horizon. Discuss the motion.

If the resistance vary as the velocity and the range on the horizontal plane through the point of projection is a maximum, show that the angle α which the direction of projection makes with the vertical is given by

$$\frac{\lambda\,(1+\lambda \cos \alpha)}{\cos \alpha+\lambda}=\log\,(1+\lambda \sec \alpha)$$

where λ is the ratio of the velocity of projection to the terminal velocity.

2. A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane. Obtain the differential equation of its path.

A particle moves with a central acceleration $\mu\,(r^5-c^4r)$, being projected from an apse at distance c with a velocity $\sqrt{\left(\frac{2\mu}{3}\right)}\,c^3$; show that its path is the curve

$$x^4+y^4=c^4.$$

3. Prove the relation $m=u-e \sin u$ between the mean anomaly m and the eccentric anomaly u of a particle moving in an ellipse of eccentricity e under a central force to a focus.

A planet is describing an ellipse about the sun as focus. Show that its velocity away from the sun is greatest when the radius vector to the planet is at right angles to the major axis of the path, and that it then is

$$\frac{2\mu ae}{T\sqrt{(1-e^2)}}$$

where $2a$ is the major axis, e the eccentricity, and T the periodic time.

4. Find an expression for the velocity at any point of a particle which slides down a rough cycloid, the axis of the curve being vertical and the vertex downwards.

If the particle starts from the point where the tangent makes an angle θ with the horizon and comes to rest at the vertex, show that

$$\mu e^{\mu\theta}=\sin\theta-\mu\cos\theta$$

μ being the coefficient of friction.

5 (*a*) Two equal centres of repulsive force are at a distance $2a$, and the law of force is $\frac{\mu}{r^2}+\frac{\mu a}{r^3}$; find the time of the small oscillation of a particle on the line joining the centres.

(*b*) A particle of mass m moves in a straight line under a force $m\mu^2$ (distance) towards a fixed point in the straight line and under a small resistance to its motion equal to $m.\mu$ (velocity) ; discuss the motion.

1964

1. (*a*) Show that the area covered by a gun on a hill of slope α is an ellipse whose focus is at the gun, eccentricity $\sin\alpha$ and area

$$\pi\,(V^3/g^2)\sec^3\alpha,$$

where V is the muzzle velocity.

(*b*) A particle starts from rest at the highest point and moves down along the outside of the arc of a smooth vertical circle. Prove that after leaving the circle the path describes a parabola whose latus rectum is $\frac{16}{27}a$, where a is the radius of the circle. (B. Sc. Dynamics)

2. A particle moves in an ellipse under a force which is always directed towards the focus. Find the law of force and the velocity at any point of its path.

A particle moves with a central acceleration

$$\mu\left(r+\frac{a^4}{r^3}\right)$$

being projected from an apse at distance a with a velocity $2\sqrt{\mu}.a$. Show that it describes the curve

$$r^2\,(2+\cos\sqrt{3}\theta)=3a^2.$$

3. Prove that in a parabolic motion under the law $\frac{\mu}{r^2}$, the time t and the angle θ measured from the axis satisfy the relation
$$t=\sqrt{\left(\frac{2a^3}{\mu}\right)}\left[\tan\frac{\theta}{2}+\frac{1}{3}\tan^3\frac{\theta}{2}\right].$$
The perihelion distance of a comet describing a parabolic path is $\frac{1}{n}$ of the radius of the Earth's path supposed circular. Show that the time that the comet will remain within the Earth's orbit is

$$\frac{2}{3\pi}\,\frac{n+2}{n}\cdot\sqrt{\left(\frac{n-1}{2n}\right)}\text{ of a year.}$$

4. A rough cycloid has its plane vertical and the line joining its cusps horizontal. A heavy particle slides down the curve from rest at a cusp and comes to rest again at the point on the other side of the vertex where the tangent is inclined at 45° to the vertical. Show that the coefficient of friction satisfies the equation

$$3\pi\mu+4\log_e(1+\mu)=2\log_e 2.$$

5. A particle is projected vertically upwards under gravity (supposed constant) in a resisting medium whose resistance varies as the square of the velocity. Find the velocity when the particle has described any distance and also the velocity at the end of any time.

A particle, of mass m, is projected vertically under gravity, the resistance of the air being mk times the velocity.

Show that the greatest height attained by the particle is

$$\frac{V^2}{g}[\lambda - \log(1+\lambda)]$$

where V is the terminal velocity of the particle and λV is its initial vertical velocity.

6. (*a*) Two bodies, of masses M and M', are attached to the lower end of an elastic string whose upper end is fixed, and hang at rest ; M' falls off ; show that the distance of M from the upper end of the string at time t is

$$a+b+c\cos\left\{\sqrt{\left(\frac{g}{b}\right)}\,t\right\},$$

where a is the unstretched length of the string, and b and c the distances by which it would be stretched when supporting M and M' respectively. (B. Sc Dynamics)

(*b*) If a pendulum oscillates in a medium the resistance of which varies as the velocity, show that the oscillations are isochronous.

1966

1. (*a*) A mass m hangs from a fixed point by a light string and is given a small vertical displacement Shew that its motion is simple harmonic.

If l is the length of a string when the system is in equilibrium and n the number of oscillations per second, shew that the natural length of the string is $l-(g/4\pi^2n^2)$.

(*b*) A particle moves along the axis of x starting from rest at $x=a$: for an interval t_1 from the beginning of the motion the acceleration is $-\mu x$, for a subsequent interval t_2 the acceleration is μx, and at the end of this interval the particle is at the origin again ; prove that

$\tan(t_1\sqrt{\mu})\tanh(t_2\sqrt{\mu})=1$. (B. Sc. Dynamics)

2. (*a*) A particle is projected with velocity U vertically upwards under gravity in a medium whose resistance varies

as the square of the velocity ; prove that the particle will return to the point of projection with a velocity

$$UV/\sqrt{(U^2+V^2)},$$

where V is the terminal velocity in the medium.

(*b*) A heavy bead of mass m slides on a smooth wire in the shape of a cycloid, whose axis is vertical and vertex upwards, in a medium whose resistance is $\mu^2v^2/2c$ and the distance of the starting point from the vertex is c ; shew that the time of descent to the cusp is

$$\sqrt{[\{8a\ (4a-c)/gc\}]},$$

where $2a$ is the length of the axis of the cycloid.

3. (*a*) A particle moves in an ellipse under a force which is always directed towards its focus ; find the law of force and the velocity at any point of its path.

(*b*) A particle moves under a force

$$m\mu\ \{3au^4-2\ (a^2-b^2)\ u^5\},$$

$a > b$, and is projected from an apse at a distance $a+b$ with velocity $\sqrt{\mu}\div(a+b)$; shew that its orbit is $r=a+b\cos\theta$.

4. (*a*) The perihelion distance of a comet describing a parabolic path is $\frac{1}{n}$ of the radius of Earth's path supposed circular ; shew that the time that the comet will remain within the Earth's orbit is

$$\frac{2}{3\pi}\cdot\frac{n+2}{n}\sqrt{\left(\frac{n-1}{2n}\right)} \text{ of a year.}$$

(*b*) A comet is moving in a parabola about the sun as focus ; when at the end of its latus-rectum its velocity suddenly becomes altered in the ratio $n:1$, where $n<1$: shew that the comet will describe an ellipse whose eccentricity is

$$\sqrt{(1-2n^2+2n^4)},$$

and whose major axis is

$$\frac{l}{1-n^2},$$

where $2l$ was the latus-rectum of the parabolic path.

5. (*a*) A particle is projected with a velocity v in a vertical plane. Shew that there are two directions of projection in order that it may pass through a given point (a, b), and that the difference of the tangents of these angles is

$$\frac{2}{ag}\sqrt{\{(v^4-2v^2bg-a^2g^2)\}}.$$

(*b*) A particle is projected at an elevation α, and after t seconds it appears to have an elevation β as seen from the point of projection. Prove that its initial velocity was

$$\frac{gt\cos\beta}{2\sin(\alpha-\beta)}.\quad \text{(B. Sc. Dynamics)}$$

5. (*a*) A heavy bead slides on a smooth fixed circular wire of radius a. If it be projected from the lowest point with a velocity just sufficient to carry it to the highest point, prove that the radius through the bead, in time t, will turn through an angle

$$2\tan^{-1}\left\{\sinh t\sqrt{\left(\frac{g}{a}\right)}\right\}.\quad \text{(B. Sc. Dynamics)}$$

(*b*) The base of a rough cycloidal arc is horizontal and its vertex downwards; a bead slides along it starting from rest at the cusp and coming to rest at the vertex. Shew that $\mu^2e^{\mu\pi}=1$, μ being the coefficient of friction.



VIKRAM UNIVERSITY PAPERS

1963

1. (*a*) A particle moves in an ellipse under a force which is always directed towards its focus ; find the law of force and the velocity at any point of its path.

(*b*) A particle describes an ellipse under a force $\frac{\mu}{(\text{distance})^2}$ towards the focus ; if it is projected with velocity v from a point distance r from the centre of force, show that its periodic time is $\frac{2\pi}{\sqrt{\mu}}\left[\frac{2}{r}-\frac{v^2}{\mu}\right]^{-3/2}$.

2. A rough cycloid has its plane vertical and the line joining its cusps horizontal. A heavy particle slides down the curve from rest at a cusp and comes to rest again at the point on the other side of the vertex where the tangent is inclined at 45° to the vertical. Show that the coefficient of friction satisfies the equation.

$$3\mu\pi+4 \log (1+\mu)=2 \log 2.$$

3. (*a*) A particle falls under gravity in a resisting medium whose resistance varies as the square of the velocity. If it starts from rest find (i) the velocity after distance s, and (ii) the distance after time t.

(*b*) A particle of unit mass is projected with velocity u at an angle α above the horizon in a medium whose resistance is k times the velocity. Show that its direction will again make an angle α with the horizon after a time

$$\frac{1}{k}\log\left[1+\frac{2ku}{g}\sin\alpha\right]$$

4. (*a*) Define hodograph and prove that the hodograph of a central orbit is a reciprocal of the orbit with respect to the centre of force S turned through a right angle about S.

(*b*) A circular tube of radius a revolves uniformly about a vertical diameter with angular velocity $\sqrt{\left(\frac{ng}{a}\right)}$, and a particle is projected from its lowest point with velocity just sufficient to carry it to the highest point. *Show that the time of describing the first quadrant is*

$$\sqrt{\left(\frac{a}{(n+1)\,g}\right)}\log\left[\sqrt{(n+2)}+\sqrt{(n+1)}\right].$$

1964

1. (*a*) A particle moves in a plane with an acceleration f which is always directed towards a fixed point O in the plane. Prove that the (p, r) equation of the path is

$$f=\frac{h^2}{p^3}\frac{dp}{dr}$$

(*b*) A particle moves with a central acceleration $\mu\left(r+\frac{2a^3}{r^2}\right)$, being projected from an apse at a distance a with twice the velocity for a circle at that distance. Find the other apsidal distance.

2. (*a*) A particle is projected with velocity u at an inclination α above the horizon in a medium whose resistance per unit mass is k times the velocity; find the equation of the path described by it.

(*b*) A particle acted on by gravity is projected in a medium of which the resistance varies as the velocity. Show that its acceleration retains a fixed direction and diminishes without limit to zero.

3. (*a*) Obtain the tangential and normal components of the acceleration of a particle moving along a plane curve.

(*b*) A particle moves in a smooth tube in the form of a catenary being attracted to the directrix by a force

proportional to the distance from it. Show that the motion is simple harmonic.

4. A particle P moves in a smooth circular tube of radius a, which turns with uniform angular velocity ω about a vertical diameter. If the angular distance of the particle at any time t from the lowest point is θ, and if it be at rest relative to the tube when $\theta=\alpha$, where $\cos\frac{\alpha}{2}=\frac{1}{\omega}\sqrt{\left(\frac{g}{a}\right)}$, then prove that at any subsequent time t

$$\cot\frac{\theta}{2}=\cot\frac{\alpha}{2}\cosh\left(\omega t\sin\frac{\alpha}{2}\right).$$

1965

1. (*a*) Show that, in every central orbit, the sectorial area traced out by the radius vector to the centre of force increases uniformly per unit of time, and the linear velocity varies inversely as the perpendicular from the centre upon the tangent to the path.

(*b*) A particle of mass m is attached to a fixed point by an elastic string of natural length a, the coefficient of elasticity being nmg; it is projected from an *apse* at a distance a with velocity $\sqrt{(2\,pgh)}$; shew that the other apsidal distance is given by the equation $nr^2(r-a)-2\,pha\,(r+a)=0$.

2. (*a*) A particle is projected vertically upwards against gravity (supposed constant) in a resisting medium whose resistance varies as the square of the velocity; find the motion if V is the velocity of projection.

(*b*) A particle moves with a central acceleration P in a medium of which the resistance is k (velocity)^2; shew that the equation to its path is

$$\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}\,e^{2ks},$$

where s is the length the of arc described, and h is twice the initial moment of momentum about the centre of force.

(*b*) A circular tube of radius a revolves uniformly about a vertical diameter with angular velocity $\sqrt{\left(\frac{ng}{a}\right)}$, and a particle is projected from its lowest point with velocity just sufficient to carry it to the highest point. Show that the time of describing the first quadrant is

$$\sqrt{\left(\frac{a}{(n+1)g}\right)} \log\left[\sqrt{(n+2)}+\sqrt{(n+1)}\right].$$

1964

1. (*a*) A particle moves in a plane with an acceleration f which is always directed towards a fixed point O in the plane. Prove that the (p, r) equation of the path is

$$f=\frac{h^2}{p^3}\frac{dp}{dr}$$

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$$\cot\frac{\theta}{2}=\cot\frac{\alpha}{2}\cosh\left(\omega t\sin\frac{\alpha}{2}\right).$$

1965

1. (*a*) Show that, in every central orbit, the sectorial area traced out by the radius vector to the centre of force increases uniformly per unit of time, and the linear velocity varies inversely as the perpendicular from the centre upon the tangent to the path.

(*b*) A particle of mass m is attached to a fixed point by an elastic string of natural length a, the coefficient of elasticity being nmg; it is projected from an *apse* at a distance a with velocity $\sqrt{(2pgh)}$; shew that the other apsidal distance is given by the equation $nr^2(r-a)-2pha(r+a)=0$.

2. (*a*) A particle is projected vertically upwards against gravity (supposed constant) in a resisting medium whose resistance varies as the square of the velocity; find the motion if V is the velocity of projection.

(*b*) A particle moves with a central acceleration P in a medium of which the resistance is k (velocity)^2; shew that the equation to its path is

$$\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}e^{2ks},$$

where s is the length of arc described, and h is twice the initial moment of momentum about the centre of force.

(*b*) A circular tube of radius a revolves uniformly about a vertical diameter with angular velocity $\sqrt{\left(\frac{ng}{a}\right)}$, and a particle is projected from its lowest point with velocity just sufficient to carry it to the highest point. Show that the time of describing the first quadrant is

$$\sqrt{\left(\frac{a}{(n+1)\,g}\right)} \log [\sqrt{(n+2)}+\sqrt{(n+1)}].$$

1964

1. (*a*) A particle moves in a plane with an acceleration f which is always directed towards a fixed point O in the plane. Prove that the (p, r) equation of the path is

$$f=\frac{h^2}{p^3}\frac{dp}{dr}$$

(*b*) A particle moves with a central acceleration $\mu\left(r+\frac{2a^3}{r^2}\right)$, being projected from an apse at a distance a with twice the velocity for a circle at that distance. Find the other apsidal distance.

2. (*a*) A particle is projected with velocity u at an inclination α above the horizon in a medium whose resistance per unit mass is k times the velocity; find the equation of the path described by it.

(*b*) A particle acted on by gravity is projected in a medium of which the resistance varies as the velocity. Show that its acceleration retains a fixed direction and diminishes without limit to zero.

3. (*a*) Obtain the tangential and normal components of the acceleration of a particle moving along a plane curve.

(*b*) A particle moves in a smooth tube in the form of a catenary being attracted to the directrix by a force

proportional to the distance from it. Show that the motion is simple harmonic.

4. A particle P moves in a smooth circular tube of radius a, which turns with uniform angular velocity ω about a vertical diameter. If the angular distance of the particle at any time t from the lowest point is θ, and if it be at rest relative to the tube when $\theta=\alpha$, where $\cos\frac{\alpha}{2}=\frac{1}{\omega}\sqrt{\left(\frac{g}{a}\right)}$, then prove that at any subsequent time t

$$\cot\frac{\theta}{2}=\cot\frac{\alpha}{2}\cosh\left(\omega t\sin\frac{\alpha}{2}\right).$$

1965

1. (*a*) Show that, in every central orbit, the sectorial area traced out by the radius vector to the centre of force increases uniformly per unit of time, and the linear velocity varies inversely as the perpendicular from the centre upon the tangent to the path.

(*b*) A particle of mass m is attached to a fixed point by an elastic string of natural length a, the coefficient of elasticity being mg; it is projected from an *apse* at a distance a with velocity $\sqrt{(2\,pgh)}$; shew that the other apsidal distance is given by the equation $nr^2\,(r-a)-2\,pha\,(r+a)=0$

2. (*a*) A particle is projected vertically upwards against gravity (supposed constant) in a resisting medium whose resistance varies as the square of the velocity ; find the motion if V is the velocity of projection.

(*b*) A particle moves with a central acceleration P in a medium of which the resistance is k (velocity)^2; shew that the equation to its path is

$$\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}\,e^{2ks},$$

where s is the length of arc described, and h is twice the initial moment of momentum about the centre of force.

3. (*a*) A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc ; shew that the time of reaching the vertex is

$$2\sqrt{\left(\frac{a}{g}\right)}\,\sin^{-1}\left[\frac{2\sqrt{(ga)}}{\sqrt{(4ag+V^2)}}\right]$$

where a is the radius of the generating circle.

(*b*) A small bead, of mass m, moves on a smooth circular wire, being acted upon by a central attraction $m\mu/(\text{distance})^2$ to a point within the circle situated at a distance b from its centre. Shew that, in order that the bead may move completely round the circle, its velocity at the point of the wire nearest the centre of force must not be less than

$$\sqrt{\left(\frac{4\mu b}{a^2-b^2}\right)}.$$

4. (*a*) A parabolic wire, whose axis is vertical and vertex downwards, rotates about its axis with uniform angular velocity ω. A ring is placed at any point of it in relative rest ; shew that it will move upwards or down-wards according as $\omega^2 \lessgtr g/(2a)$, and will remain at rest if $\omega^2=g/(2a)$, where $4a$ is the latus-rectum of the parabola.

(*b*) Shew that the hodograph of a circle described under a force to a point on the circumference is a parabola.

1966

1. (*a*) Enunciate Kepler's three laws for planetary orbits and deduce briefly that Newton's law of Gravitation is true throughout the solar system.

(*b*) In a central orbit the force is $\mu u^3\,(3+2a^2u^2)$; if the particle be projected at a distance a with a velocity $(5\mu/a^2)^{1/2}$ in a direction making $\tan^{-1}\frac{1}{2}$ with the radius, shew that the equation to the path is

$$r=a\tan\theta.$$

2. (*a*) A particle moves in a resisting medium with a given central acceleration P; the path of the particle being given, shew that the resistance is

$$-\frac{1}{2p^2}\frac{d}{ds}\left[p^3\frac{dr}{dp}P\right].$$

(*b*) A bead moves on a smooth circular wire in a vertical plane under a resistance {$=K$ (velocity)^2}; find the motion.

3. (*a*) The base of a rough cycloidal arc is horizontal and its vertex downwards; a bead slides along it starting from rest at the cusp and coming to rest at the vertex.

$$\mu^2 e^{\mu\pi}=1.$$

(*b*) A curve in a vertical plane is such that the time of describing any arc, measured from a fixed point O, is equal to the time of sliding down the chord of the arc. Show that the curve is a Lemniscate of Bernoulli, whose node is at O and whose axis is inclined at 45° to the vertical.

4. (*a*) Show that the hodograph of a central orbit is a reciprocal of the orbit with respect to the centre of force S turned through a right angle about S.

(*b*) A smooth plane tube, revolving with angular velocity w about a point O in its plane, contains a particle of mass m, which is acted upon by a force mw^2r towards O; shew that the reaction of the tube is $A+\frac{B}{\rho}$, where A and B are constants and ρ is the radius of curvature of the tube at the point occupied by the particle.

RAJASTHAN UNIVERSITY PAPERS

1963

1. A particle moves under the action of a central force $\mu\,(u^5-\frac{1}{8}a^2u^7)$, the velocity of projection being $(25\mu/8a^4)^{1/2}$ and the angle of projection $\sin^{-1}\frac{4}{5}$. Prove that the polar equation of the path is

$$3a^2=(4r^2-a^2)\,(\theta+c)^2,$$

where $u=1/r$ and c is a constant of integration.

2. Determine the motion of a particle in a smooth plane tube which is rotating about a point in its plane.

A particle is free to move along a smooth curve $r^n=a^n \cos n\theta$ which revolves in its own plane with uniform angular velocity ω. Prove that if the particle be initially at rest at $\theta=0$, the distance from origin at any time afterwards is $$a \operatorname{sech}^{1/n} \omega t.$$

3. A particle slides down a cusp down the arc of a rough cycloid, the axis of which is vertical. Prove that its velocity at the vertex will bear to the velocity at the same point when the cycloid is smooth in the ratio

$$(e^{-\mu\pi}-\mu^2)^{1/2} : (1+\mu^2)^{1/2},$$

μ being the coefficient of friction.

4. A particle is projected horizontally from the lowest point of a smooth elliptic arc whose major axis $2a$ is vertical and moves under gravity along the concave side. Prove that it will leave the curve at some point if the velocity of projection V lies between $2ga$ and $ga\,(5-e^2)$; and if the velocity have the latter value, prove that the particle will continue to move round the ellipse in time

$$2\left(\frac{a}{g}\right)^{1/2}\int_0^{\pi}\frac{(1-e^2\cos^2\phi)\,d\phi}{(3-e^2+2\cos\phi)^{1/2}}.$$

1964

1. (*a*) A particle moves in a plane with an acceleration which is always directed to a fixed point in a plane. Find the differential equation of the path

(*b*) A particle moves with a central acceleration $\mu\left(r+\frac{a^4}{r^3}\right)$ being projected from an apse at a distance a with a velocity $2a\sqrt{\mu}$; show that it describes the curve

$$r^2(2+\cos\sqrt{3}\theta)=3a^2.$$

2. (*a*) A particle is projected along the inner surface of a rough sphere and is acted on by no forces; show that it will return to the point of projection at the end of time

$$\frac{a}{\mu V}(e^{2\mu\pi}-1);$$

where a is the radius of the sphere, V is the velocity of projection and μ is the coefficient of friction.

(*b*) The base of a cycloidal arc is horizontal and its vertex downwards, a bead slides along it starting from rest at the cusp and coming to rest at the vertex. Show that

$$\mu^2 e^{\mu\pi}=1.$$

3. (*a*) Discuss the motion of a bead constrained to slide under given forces on a curve revolving in its own plane about a given fixed point, and deduce the procedure of reducing such a revolving curve to rest.

(*b*) A bead can move on a smooth circular wire and is initially at rest at a point A. The wire is made to rotate uniformly, in its own plane, with unit angular velocity, about the other end of the diameter through A. Show that the pressure between the bead and the wire vanishes at a time $\log_e\{(3+\sqrt{5})/2\}$ after the start.

4. (*a*) If the resistance vary as the velocity and the range on the horizontal plane through the point of projection is a maximum, show that the angle α which the direction of projection makes with the vertical is given by

$$\lambda\,(1+\lambda\cos\alpha)=(\cos\alpha+\lambda)\log_e(1+\lambda\sec\alpha),$$

where λ is the ratio of the velocity of projection to the terminal velocity.

(*b*) If a particle of mass m be acted upon by equal constant forces mf in the directions of the tangent and normal to its path, and if the resistance be mk^2v^2f, prove that the intrinsic equation of the path is

$$e^{2k^2fs}-1=k^2u^2\,(e^{2\psi}-1),$$

where u is the velocity of projection.

1965

1. (*a*) A particle is constrained to move on a rough plane curve under given forces in the plane of the curve, determine the motion.

(*b*) A particle under no forces is projected with velocity V in a rough tube in the form of an equiangular spiral at a distance a from the pole. Show that it will arrive at the pole in time

$$\frac{a}{V}\cdot\frac{1}{\cos\alpha-\mu\sin\alpha},$$

α being the angle of the spiral and μ ($<\cot\alpha$) the coefficient of friction.

2. (*a*) Obtain equations of motion under any system of forces for a particle constrained to move on a smooth curve, which lies in a horizontal plane and turns with constant angular velocity about a fixed vertical axis.

(*b*) A bead, free to slide along a smooth wire in the shape of the cardioid $r=a\,(1+\cos\theta)$, which lies in a

horizontal plane, is placed at rest at the apse. The wire is then set in rotation with constant angular velocity about a vertical axis passing through the cusp. Show that in subsequent motion, the bead always moves towards but never reaches the cusp.

3. (*a*) Prove that a heavy particle moving in a resisting medium satisfies the equation

$$\frac{d^2y}{dx^2}=-\frac{g}{u^2},$$

where OY is vertical and u is the horizontal component of the velocity.

(*b*) A particle moves in a resisting medium with a given central acceleration P; the path of the particle being given, show that the resistance is

$$-\frac{1}{2p^2}\frac{d}{ds}\left(p^3\frac{dr}{dp}P\right).$$

4. (*a*) Find the differential equation of the path of a particle moving in a plane with an acceleration which is always directed to a fixed point in the plane.

(*b*) A particle moves with a central acceleration $\mu\left(\frac{5}{r^3}+\frac{8c^2}{r^5}\right)$ being projected from an apse at a distance c with velocity $\frac{3\sqrt{\mu}}{c}$. Prove that the orbit is

$$r=c\cos\frac{2\theta}{3}.$$

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AGRA UNIVERSITY PAPERS

1963

1. Obtain the equation

$$\left(\frac{d^2u}{d\theta^2}+u\right) h^2u^2=F,$$

for a central orbit.

A particle, subject to a central force per unit mass equal to

$$\mu\,[2\,(a^2+b^2)\,u^5-3a^2b^2u^7],$$

is projected at the distance a with a velocity $\frac{\sqrt{\mu}}{a}$ in a direction at right angles to the initial distance. Show that the path is the curve

$$r^2=a^2\cos^2\theta+b^2\sin^2\theta.$$

2. *(a)* Explain the terms True, Eccentric and Mean Anomalies.

(b) If the eccentricity e of the earth's orbit round the sun is so small that its cubic and higher powers can be neglected, show that

$$\theta=nt+2e\sin nt+\tfrac{5}{4}e^2\sin 2nt$$

and $$r=a\,[1-e\cos nt+\tfrac{1}{2}e^2\,(1-\cos 2nt)],$$

where r is the distance of the earth from the sun when the true anomaly is θ and mean anomaly nt.

3. A particle is constrained to move on a rough plane curve under given forces in the plane of the curve ; determine the motion.

A particle, under no forces, is projected with velocity V in a rough tube in the form of an equiangular spiral at a distance a from the pole and towards the pole. Show that it will arrive at the pole in time

$$\frac{a}{V}\cdot\frac{1}{\cos\alpha-\mu\sin\alpha},$$

α being the angle of the spiral and μ ($< \cot \alpha$) the coefficient of friction.

4. Investigate the motion of a heavy particle in a medium the resistance of which varies as the velocity.

If the resistance vary as the velocity and the range on the horizontal plane through the point of projection is a maximum, show that the angle α which the direction of projection makes with the vertical is given by

$$\frac{\lambda (1+\lambda \cos \alpha)}{\cos \alpha+\lambda}=\log (1+\lambda \sec \alpha),$$

where λ is the ratio of the velocity of projection to the terminal velocity.

5. *(a)* Prove that the hodograph of a catenary, described freely under an acceleration parallel to its axis, is a straight line described with velocity proportional to that in the catenary.

(b) A particle moves in a smooth plane tube in the form of a parabola and the tube is initially at rest with the particle at the vertex and then suddenly begins to rotate about the focus with uniform angular velocity. Prove that pressure in any position is proportional to $\sqrt{r}\,(3\sqrt{a}-4\sqrt{r})$, where r is the distance from the focus and $4a$ is latus rectum.

1964

1. If the law of force be

$$\mu \left(u^4 - \frac{10}{9}\, au^5\right)$$

and the particle be projected from an apse at a distance $5a$ with a velocity equal to $\sqrt{(\frac{5}{7})}$ of that in a circle at the same distance, show that the orbit is the limacon

$$r=a\,(3+2 \cos \theta).$$

2. Find the time of description of a given arc of a parabolic orbit starting from the vertex, when the law of attraction is that of the inverse square of the distance.

The perihelion distance of a comet describing a parabolic path is $\frac{1}{n}$ of the radius of the Earth's path, supposed circular Show that the time that the comet will remain within the Earth's orbit is

$$\frac{2}{3\pi}\cdot\frac{n+2}{n}\cdot\sqrt{\left(\frac{n-1}{2n}\right)}$$

of a year.

3. *(a)* Discuss the motion of a particle on a rough curve under gravity

(b) A bead is constrained to move on a smooth wire in the form of an equiangular spiral. It is attracted to the pole of the spiral by a force, $m\mu$ (distance)$^{-2}$, and starts from rest at a distance b from the pole. Show that, if the equation to the spiral be $r=ae^{\theta \cot \alpha}$, the time of arriving at the pole is

$$\frac{\pi}{2}\sqrt{\left(\frac{b^3}{2\mu}\right)} \sec \alpha.$$

Find also the reaction of the curve at any instant.

4. A bead moves on a smooth wire in a vertical plane under a resistance equal to k (velocity)^2 ; find the motion.

A heavy bead, of mass m, slides on a smooth wire in the shape of a cycloid, whose axis is vertical and vertex upwards, in a medium whose resistance is

$$m\frac{v^2}{2c}$$

and the distance of the starting point from the vertex is c ; show that the time of descent to the cusp is

$$\sqrt{\left(\frac{8a\,(4a-c)}{gc}\right)}.$$

where $2a$ is the length of the axis of the cycloid.

5. *(a)* The resistance of the air being supposed to vary as the cube of the velocity, show that the hodograph of a projectile is $x^3+3xy^2=ay^3+b$, the axis of x being vertical

(b) A particle moves on a wire in the form of a circle of radius a, constrained to rotate in its own plane with angular velocity ω about a point O of itself. The particle is initially at the end of the diameter through O and is projected with velocity $2b\omega$, relative to the wire. Show that the particle describes a quadrant of the circle in time

$$\frac{\pi a}{4b\omega}\left[1+\frac{a^2}{4b^2}\left(1-\frac{2}{\pi}\right)\right],$$

where b is large.

1965

1. A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane. Obtain the equation of motion as

$$P=\frac{h^2}{p^3}\frac{dp}{dr},$$

where P, p and h have the usual meaning.

Show that the only law for a central attraction, for which the velocity in a circle at any distance is equal to the velocity from infinity to that distance, is that of the inverse cube.

2. A particle describes a parabola under a force to the focus; find the time of describing an arc of the parabola.

If the parabolic orbits of two comets intersect the orbit of the earth, supposed circular, in the same two given points, and if t_1, t_2 be the times in which the comets respectively move from one of these points to the other, prove that

$$(t_1+t_2)^{2/3}+(t_1-t_2)^{2/3}=\left(\frac{4}{3\pi}\right)^{2/3},$$

the unit of time being a year.

3. (*a*) A particle slides in a vertical plane down a rough cycloidal arc whose axis is vertical and vertex downwards, starting from a point where the tangent makes an angle θ with the horizon and coming to rest at the vertex. Show that

$$\mu e^{\mu\theta}=\sin\theta-\mu\cos\theta.$$

(*b*) A bead slides down a rough circular wire, which is in a vertical plane, starting from rest at the end of a horizontal diameter. When it has described an angle θ about the centre, show that the square of its angular velocity is

$$\frac{2g}{a(1+4\mu^2)}[(1-2\mu^2)\sin\theta+3\mu(\cos\theta-e^{-2\mu\theta})],$$

where μ is the coefficient of friction and a the radius of the circle.

4. A particle is projected upwards in a resisting medium whose resistance varies as the square of the velocity. Find the motion.

A heavy particle is projected vertically upwards with velocity u in a medium, the resistance of which is $gu^{-2}\tan^2\alpha$ times the square of the velocity, α being a constant. Show that the particle will return to the point of projection with velocity $u\cos\alpha$ after a time

$$ug^{-1}\cot c\left(\alpha+\log\frac{\cos\alpha}{1-\sin\alpha}\right).$$

5. (*a*) If a particle describes a lemniscate under a force to its pole, show that the equation to the hodograph is

$$r^2=a^2\sec^3\frac{\pi-2\theta}{3}.$$

(*b*) A smooth circular tube contains a particle of mass m and lies on a smooth table. The tube starts rotating with constant angular velocity ω about an axis perpendicular to the plane of the tube which passes through the other end O of the diameter through the initial posi-

tion A of the particle. Show that in time t the particle will have described an angle ϕ about the centre of the tube equal to

$$4 \tan^{-1}\left(\tanh \frac{\omega t}{2}\right).$$

Show also that the reaction between the tube and the particle is then equal to

$$2ma\omega^2 \cos \frac{\phi}{2}\left(3 \cos \frac{\phi}{2} - 2\right).$$

1966

1. (*a*) A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane. Obtain the differential equation of its path in the polar form.

(*b*) A particle moves with a central acceleration,

$$= \frac{\mu}{(\text{distance})^5},$$

and is projected from an apse at a distance a with a velocity equal to n times that which would be acquired in falling from infinity. Show that the other apsidal distance is

$$\frac{a}{\sqrt{(n^2-1)}}.$$

2. (*a*) A particle describes a parabola under a force to the focus; find the time of describing an arc of the parabola.

(*b*) A planet is describing an ellipse about the sun as focus; show that its velocity away from the sun is greatest when the radius vector to the planet is at right angles to the major axis of the path, and that it then is

$$\frac{2\pi ae}{T(1-e^2)^{1/2}},$$

where $2a$ is the major axis, e the eccentricity, and T the periodic time.

3. (*a*) A bead is constrained to move on a smooth wire in the form of an equiangular spiral. It is attracted to the

pole of the spiral by a force, $=m\mu$ (distance)$^{-2}$, and starts from rest at a distance b from the pole. If the equation to the spiral be $r=ae^{\theta \cot \alpha}$, show that the time of arriving at the pole is

$$\frac{\pi}{2}\sqrt{\left(\frac{b^3}{2\mu}\right)}.\sec \alpha.$$

(*b*) The base of a rough cycloidal arc is horizontal and its vertex downwards; a bead slides along it starting from rest at the cusp and coming to rest at the vertex. Show that $\mu^2.e^{\mu\pi}=1$, where μ is the coefficient of friction.

4. (*a*) A particle falls under gravity (supposed constant) in a resisting medium whose resistance varies as the square of the velocity. Find the motion if the particle starts from rest.

(*b*) A particle of mass m is projected vertically under gravity, the resistance of the air being mk times the velocity. Show that the greatest height attained by the particle is

$$\frac{V^2}{g}\left[\lambda-\log\left(1+\lambda\right)\right],$$

where V is the terminal velocity of the particle and λV is its initial vertical velocity.

5. (*a*) A particle describes an equiangular spiral about a centre of force at the pole. Show that its hodograph is also an equiangular spiral.

(*b*) A tube in the form of the cardioid $r=a\,(1+\cos\theta)$ is placed with its axis vertical and cusp uppermost, and revolves round the axis with angular velocity

$$\sqrt{\left(\frac{g}{a}\right)}.$$

A particle is projected from the lowest point of the tube along the tube with velocity $\sqrt{(3ga)}$. Show that the particle will ascend until it is on a level with the cusp.

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